Portfolio optimization with short-selling and spin-glass

Lorenzo Bongini¹, Mirko Degli Esposti², Cristian Giardinà², and Augusto Schianchi³

¹ Istituto Nazionale di Fisica della Materia, Largo E. Fermi 2, 50125 Firenze, Italy
² Dipartimento di Matematica dell' Università di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy
³ Dipartimento di Economia dell' Università di Parma, via Kennedy 6, 43100 Parma, Italy

the date of receipt and acceptance should be inserted later

Abstract. In this paper, we solve a general problem of optimizing a portfolio in a futures markets framework, extending the previous work of Galluccio, Bouchaud and Potters [Physica A 259, 449 (1998)]. We allow for long buying/short selling of a relatively large number of assets, assuming a fixed level of margin requirement. Because of non-linearity in the constraint, we derive a multiple equilibrium solution, in a size exponential respect to the number of assets. That means that we can not obtain the unique efficiency frontier, but many of them and each one is related to different levels of risk. Such a problem is analogous to that of finding the ground state in long-ranged Ising spin glass with external field. In order to get the best portfolio (i.e. that is along the best efficiency frontier), we have to implement a two-step procedure, performing the exhaustive enumeration of all local minima. We develop a concrete application, where the different part of the proposed solution are computed.

PACS. 89.65.Gh Economics, businnes, and financial market

1 Introduction

Portfolio theory is a basic pillar in economic analysis. It was originally proposed by Harry Markovitz [1] during the 50's. The approach was that the return of any financial activity is described by a random variable, whose expected mean (measure for reward) and variance (interpreted as volatility) are assumed to be known from its historical past. The selection of a particular portfolio is based on the mean-variance principle: i.e., if two portfolios are given, and the expected return of the first portfolio is higher than the second one, or the variance of the first portfolio is lower than the second one, we say that the first portfolio dominates the second; the latter being outside the
decision field of a rational investor. Portfolio selection allows us to find the set of efficient portfolios, i.e., those portfolios not dominated by anything else. The rational investor eventually chooses among these efficient portfolios, in a subjective manner, according to his preferences towards risk. Much criticism, over the years, has been addressed to Markowitz’s model. For instance, the choice of variance as a signal of risk has been criticized on logical grounds: the model requires a quadratic utility function and set of returns that must be normally distributed, but empirical analyses of financial price-data show that short time variations of the price of different assets deviate from the Gaussian distribution [2–4], which would be expected if agents were trading indipendently. Criticism has also been made of other aspects of the original model.

In spite of this, the Markowitz approach has been very successful, because of its ability to grasp the hard core of the problem of how to allocate wealth among alternative assets. Extended versions of the model (for instance, in order to increase the number and types of assets considered; or to reduce computations for the estimated matrix of variances/covariances among returns), have almost continuously been introduced, and today various and updated versions of the original model are widely used by financial practitioners (for instance: [5,6]). Quoting from a successful fund management story 1:

Even if point estimates of risk and return variables fail to represent reality fairly, insofar as inputs stem from well-grounded interrelationships, the mean-variance optimization process produces valuable insight into efficient portfolio alternatives.

A very interesting extension of the original model to the "short sales" case is due to the contribution of Lintner [7,8]. In fact, in this paper we move in the same direction extending and generalizing the traditional analysis (short-selling included), by considering the case of futures markets, where the short sale problem is regulated through the mechanism of margin accounts. Specifically - regarding the current state of the art in portfolio selection - the model for futures markets allows for:

- long-buying/short selling activities in equities,
- leveraging on margin accounts,
- a set of a (relatively) large number of assets to be prospected.

The main result we aim to discuss here is that the portfolio optimization problem in futures markets naturally belongs to the realm of "complex" problems. In particular, a very large number of (quasi) equilibrium solutions coexists in the model and any procedure developed to reach a decision regarding the structure of the portfolio must face this problem. To be a little bit more precise, if we have \(N\) assets available to the investor, for a fixed expected return \(R\), the number \(n(N,R)\) of risk local minima grows exponentially in the number of assets, i.e.:

\[
n(N, R) \sim e^{\alpha(R)N}
\]

---
where $\omega(R)$ is a positive number depending on the portfolio return. Since we have this multiple equilibrium solution (however let us notice that the equilibria are not equivalent among themselves respect to the level of risk), we have to implement a second step in the solution procedure in order to get the global equilibrium. This conclusion is a direct consequence of the application of Lagrange optimization and the non-linear constraint on the total wealth in futures markets. The issue of non-unique equilibrium is a well-known chapter of economic analysis, particularly in models with money, increasing returns and imperfect competition [9] and a new line of research on equilibrium beliefs is being developed. However it should be emphasized that, in this paper, the issue of multiple equilibrium solutions is discussed in a quite different setting. The underlying idea is that the selection of a unique optimum portfolio obtained from portfolio optimization in its current form [5,6], works well only if we introduce drastic and perhaps unrealistic simplification. Enlarging the picture to a certain extent, by relaxing the most restrictive assumptions in line with the practical experience of fund management, when calculating the solution for the optimum portfolio, we reach rapidly the threshold of a so-called NP problem, an area of research currently still largely ignored in economic computation [10,11]. We will deal in this paper with the complexity of the problem for a concrete case of a portfolio made of 16 risky assets. We will explicitly show where complexity does arises and we will suggest the necessity of using algorithms typical of the realm of other optimization problem (like “simulated annealing” [12]).

Our analysis is built on a seminal idea by Galluccio, Bouchaud and Potters [13], who have re-stated the problem of a solution to the portfolio optimization problem in futures markets in terms of a spin glass problem (see also [14–16]). Motivated initially by the desire to understand the strange behaviour of certain magnetic alloys, the theory of spin glasses has provided a powerful paradigm for complex systems having competition and conflicting internal constraints. This is due to the fact that techniques developed for spin glasses has been successfully applied to other fields, like optimization problems, biology, social sciences [17,18]. In ref. [13], Galluccio et al. showed that the problem of portfolio selection with short-selling, regulated through the mechanism of margin accounts, is basically equivalent to that of finding the ground state of a long-ranged Ising spin glass. The correlation matrix between assets return is related to the coupling matrix of the spin glass. Arguing that correlation matrix can be regarded as a generic realization of a matrix taken from an appropriate random ensemble, they derived the exponential increase of optimal solutions with the number of assets from the well-know increase of 1-flip-stable states with the number of spins [19,20]. Here we put forward their analysis, by explicitly constructing the efficient frontier of the portfolio. To achieve this aim, when performing the minimization of risk under the non-linear constraint, we need to fix the average return of the portfolio. In the spin glass language this is the analogous of imposing an external field [21].
The rest of the paper goes as follows. In the next Section we introduce the model of portfolio optimization in futures markets and we explain the non-linear constraint that is involved. A first analysis of the portfolio variance in the spirit of ref. [13] is also presented, showing the complexity of the problem, compared to the classical Markowitz problem, where the wealth constrain is linear and one ends up always with a unique solutions. In Section 3 we generalize the optimization procedure by considering a fixed portfolio return. The general procedure and analytical calculation needed to construct the efficient frontier are obtained and similarities with the spin glass problem are enlightened. We deal with a concrete example in Section 4, considering a portfolio of 16 assets traded on the Nasdaq Market and solving the portfolio-problem by means of computer calculations. We show the multiple equilibrium solutions and we discuss the distribution of local minimum risks. Then, we calculate the efficient frontier and discuss a related averaged frontier. This last curve, as discussed in the paper, represents the decision that an investor will mostly likely taken by searching the optimal solution with standard methods of combinatorial optimization. We will also reconstruct and discuss the frontier corresponding to the ”worst of the best decisions”, namely the local minimum with higher risk (at fixed return). As we will see, the risk function has an exponential number of local minima and one needs to select by hand the lowest one. Moreover, there is the possibility that portfolios completely different among themselves correspond almost to the same value of the risk. Finally, in Section 5 some remarks on variances/covariances matrix are presented and summarizing Conclusions will follow as usual.

2 The model

In this section, we present a model of portfolio optimization, where some traditional assumptions are relaxed. Specifically, in order to allow for the maximum flexibility in the model, a hedge-fund as rational investor agent is considered. Let us start from the standard definitions. We consider a portfolio \( \mathcal{P} \) of \( N \) risky assets indexed by the subscript \( i \) which takes values \( i = 1, \ldots, N \). The variance (i.e. risk) of the portfolio is

\[
\sigma^2_p = \sum_{i,j=1}^{N} C_{ij} p_i p_j = p^T C p \tag{1}
\]

and the mean (i.e. expected average return)

\[
R_p = \sum_{i=1}^{N} p_i r_i = p^T r \tag{2}
\]

where \( p = (p_1, p_2, \ldots, p_N) \) are the shares of a given total wealth to be invested, and the external parameters are

\[ r_i \cdot \text{the expected return on asset } i, \]

\[ C_{ij} \cdot \text{the matrix of variances } (i = j) \text{ and covariances } (i \not= j). \]

and we have introduced the usual vectorial notation with \( T \) to denote the transposed.

The aim of the optimization is to find the most efficient investment strategy, i.e. to evaluate proportions \( p \) of the total wealth \( W \) that minimize the risk \( \sigma^2_p \), for a given return \( R_p \) (or viceversa shares \( p \) that maximize the return \( R_p \) for a given level of risk \( \sigma^2_p \)). In other words, we would
like to calculate the efficient frontier, that represents the relationship between the risk of the portfolio and the expected return of the portfolio itself having the best utility for the investor. Knowing the efficient frontier, once that a particular expected rate of return has been identified, we can determine the correspondent efficient portfolio; it is such that its variance (or riskiness) is a minimum. Therefore, once one of the two (either the risk or the return) has been chosen by the investor, the other variable is derived as a consequence.

Since long-buying and short-selling are allowed, and leveraging on margin accounts works, then the budget constraint is

\[ \sum_{i=1}^{N} \Gamma |p_i| = W \tag{3} \]

where \( \Gamma \) is the (fixed) margin constraint and \( p_i > 0 \) or \( p_i < 0 \), depending on the sign of the contract (buy or sell respectively). The margin is assumed fixed for all operations, and it does not change over time according to price variations of the underlying assets. Moreover, the problem of issuing futures on behalf of the financial institution is not considered. Without loss of generality we can set \( W/\Gamma = 1 \), so that, introducing the vector \( s \) whose components are \( s_i = \text{sign}(p_i) \), the budget constrain becomes:

\[ \sum_{i=1}^{N} |p_i| = p^T s = 1 \tag{4} \]

where \( \text{sign} \) denotes the sign function, \( \text{sign}(x) = 1 \) if \( x > 0 \) and \( \text{sign}(x) = -1 \) if \( x < 0 \).

### 2.1 "Complexity" of the model

Let us consider the problem of finding the minimum of the variance subjected to the only budget constraint (no fixed average portfolio return). In other words we want to minimize the portfolio variance Eq.(1) with the non-linear constraint given by Eq.(4). We introduce a Lagrangian function with one Lagrange multiplier \( \mu \):

\[ L(p, \mu) = p^T C p - \mu(p^T s - 1) \tag{5} \]

Differentiating with respect to the \( N + 1 \) unknowns \( p \) and \( \mu \) we obtain the following equations for the extreme points

\[ p = \frac{1}{2} \mu C^{-1} s \tag{6} \]

\[ p^T s = 1 \tag{7} \]

where \( C^{-1} \) is the inverse of the correlations matrix.

Inserting Eq.(6) in Eq.(7) we can solve for \( \mu \) and then for \( p \)

\[ \mu = \frac{2}{s^T C^{-1} s} \tag{8} \]

\[ p = \frac{1}{s^T C^{-1} s} C^{-1} s \tag{9} \]

Applying the \( \text{sign} \) function to both sides of the last Equation, we finally obtain

\[ s = \text{sign}(C^{-1} s) \tag{10} \]

where we have used the fact that, since \( C \) is a positive definite matrix, the same is true for \( C^{-1} \), so that \( s^T C^{-1} s > 0 \) for every value of \( s \).

The original problem has thus been mapped in finding the solution of Eq. (10): once the \( s_i \) that solve this Equation are known, the shares \( p_i \) can be calculated using Eq.
(9), while the portfolio variance is given by

$$\sigma_P^2 = \frac{1}{s^T C^{-1} s}$$  \hspace{1cm} (11)

But solving Eq. (10) is a very tough task. It is exactly the same equation that appears in spin glass theory when one looks for 1-flip stable configurations at zero temperature. It is well-known [19,20] that Eq. (10) admits for a generic random matrix $C^{-1}$ an exponential number of solution. Moreover these solutions are ”chaotic”, i.e. they are completely different one from another and they completely change varying the number of degree of freedom. In our case the matrix $C^{-1}$ is not a priori random but it is constructed from the historical dates. Nevertheless, since historical prices/returns movements are generated by market fluctuations, the correlation matrix $C$ (and so its inverse $C^{-1}$) can be seen as a generic realization of some specific random matrix ensemble (see Section 5). In this way we can borrow the results from physics and directly draw some first conclusions [13]:

- At variance with the classical Markowitz portfolio problem, where we always find a minimization equation that admits a unique solution, in the present case of futures markets we have an exponential number of portfolios for which the risk function has a (local) minimum. So we face the embarrassment of which solution to choose and we need to calculate by hand the portfolio variance on each solution to find the true minimum.

- We can have very different portfolios corresponding to (local) risk minima having almost the same risk value.

- Adding one asset to the portfolio it radically change the shape of efficient investments.

\section{3 Constructing the efficient frontier}

In the previous Section we have shown how the complexity of the problem naturally arise in the minimization procedure for the case of the global minimum of the portfolio risk. This does not say nothing about the efficient frontier (even if it shows the instability of rational investment decisions). To completely solve the problem, we have to repeat the minimization of the variance fixing the average return to the value $R$ with an extra Lagrange multiplier $\nu$.

Thus the problem is now to minimize Eq. (1) subject to Eq. (4) and to the additional constraint

$$R_P = p^T r = R$$  \hspace{1cm} (12)

We introduce the Lagrangian function

$$L(p,\mu) = p^T C p - \mu (p^T s - 1) - \nu (p^T r - R)$$  \hspace{1cm} (13)

Differentiating with respect to the $N + 2$ unknowns $p, \mu$ and $\nu$ we obtain

$$p = \frac{1}{2} \mu C^{-1} s + \frac{1}{2} \nu C^{-1} r$$

$$p^T s = 1$$

$$p^T r = R$$  \hspace{1cm} (14)

Inserting the first equation in the second and the third, we can solve for $\mu$ and $\nu$, and then for $p$. Defining

$$\alpha = s^T C^{-1} s$$
\[ \beta = r^T C^{-1} s \]
\[ \gamma = r^T C^{-1} r \]

we obtain the following expressions:
\[ \mu = 2 \frac{\gamma - R \beta}{\alpha \gamma - \beta^2} \]
\[ \nu = 2 \frac{R \alpha - \beta}{\alpha \gamma - \beta^2} \]
\[ p = \gamma - R \beta \frac{C^{-1} s}{\alpha \gamma - \beta^2} + \frac{R \alpha - \beta}{\alpha \gamma - \beta^2} C^{-1} r \]

Applying the \textit{sign} function to both sides of the last Equation and remembering that \( s = \text{sign}(p) \) by definition, we finally obtain
\[ s = \text{sign} \left( \gamma - R \beta \frac{C^{-1} s}{\alpha \gamma - \beta^2} + \frac{R \alpha - \beta}{\alpha \gamma - \beta^2} C^{-1} r \right) \]

This is the basic equation that substitute Eq.(10) in the case of a fixed average return \( R \). If we now identify the
\[ \frac{\gamma - R \beta}{\alpha \gamma - \beta^2} C^{-1} \Leftrightarrow J \]
\[ \frac{R \alpha - \beta}{\alpha \gamma - \beta^2} C^{-1} r \Leftrightarrow h \]

we establish a perfect analogy between the Equation to be solved for the portfolio optimization problem in futures markets and the one for the spin glass problem of finding the local energy minima at zero temperature in presence of an external field \( h \). Note that in Eq. (17) the coupling matrix depends on the spin configuration itself.

The general procedure for tracing the \( N \)-stocks efficient frontier in the case of \textit{futures markets} thus can be summarized as follows:

1. Fix a certain value of the average expected portfolio return \( R \).

2. For this return \( R \) solve the system of \( N \) equations (17) for the vector \( s = (s_1, s_2, \ldots, s_N) \). In general the number of solutions \( n \) will be exponential in number of assets \( N: n \sim e^{\omega N} \), where the exponential rate \( \omega = \omega(R) \) depends on the fixed return \( R \).

3. Calculate the value of the proportions investment \( p = (p_1, p_2, \ldots, p_N) \) corresponding to each solution of step 2 through formula (16) and then the associated risk. Select the lowest value of the risk and the corresponding optimum portfolio investment.

4. Increase the return \( R \) by a certain (constant) amount and repeat the entire procedure from step 2 through 4.

4 A \textbf{worked example}

In this Section we explicitly treat an example with real data: we demonstrate the “complexity” of our problem and we calculate the \textit{efficient frontier} by means of computer calculations.

4.1 \textbf{Data}

We considered the case of a portfolio consisting of \( N = 16 \) risky assets. These risky assets are some common stocks traded on the Nasdaq, in the period October 1, 1998 - November 13, 2000. An historical record of daily prices of these stocks for the \( T = 553 \) trading days of the period was used to estimate the relevant parameters - the mean return \( r_i \) and the variance/covariance matrix \( C_{ij} \). The data source is \textit{DataStream}. Calling \( x(i,k) \) the price
of the $i$-th asset (where $i = 1, \ldots, N$) at the $k$-th day
(where $k = 1, \ldots, T$), the daily rates of return are:

$$r(i, k) = \frac{x(i, k + 1) - x(i, k)}{x(i, k)}$$

(18)

while the formula used to estimate average returns and
covariances are:

$$r_i = \frac{1}{T - 1} \sum_{k=1}^{T-1} r(i, k)$$

(19)

$$C_{ij} = \frac{1}{T - 1} \sum_{k=1}^{T-1} [r(i, k) - r_i][r(j, k) - r_j]$$

(20)

The estimated mean returns are given in Table (1), which
also lists the stocks by name, while the variance/covariance
matrix is split in Tables (2) and (3).

### 4.2 Multiple solutions

First of all we analyzed the whole set of possible choices
that one obtains for a fixed value of the return on the
portfolio composed by $N = 16$ assets. We choose to fix the
portfolio return to the value of the average daily Nasdaq
index return in the period we considered: $R = R_{NAS} = 0.0014$ (i.e. 0.14%). Applying the solving technique
described in the previous Section, we found the solutions
$\hat{s} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_N)$ of the Eq. (17) doing an exhaustive enu-
meration of all the $2^N$ possible values of the $N$-dimensional
vector $s$. We found 6675 $\hat{s}^{(j)}$ vectors that satisfy Eq.(17).

From then we calculate the corresponding proportions
$p^{(j)}$ using Eq.(16) and the risk values $\hat{\sigma}^{(j)}$ using Eq.(1).

Here the index $j$ labels all the local risk minima, $j = 1, 2, \ldots, 6675$. The risk ranges between $\hat{\sigma}_{MIN} \approx 0.0090$
and $\hat{\sigma}_{MAX} \approx 0.0162$ and has an averaged risk $\hat{\sigma}_{AVE} \approx 0.0100$, where the averaged value is obviously defined as

$$\hat{\sigma}_{AVE} = \frac{1}{6675} \sum_{j=1}^{6675} \hat{\sigma}^{(j)}$$

We show how the risk values are distributed in Fig,(1),
where we plot their probability density function, i.e. the
histogram normalized to have area 1. We see that most
of the risk local minima have a risk value higher than the
global risk minimum $\hat{\sigma}_{MIN}$.

It is important to remark that the local minimum which
corresponds to the maximum of the distribution shown
in Fig. (1) are in fact the solutions that one would ob-
tain with very ”high probability”. To be more precise, if
one consider portfolios of some bigger dimension, just say
100 assets, then it would be impossible to compute ex-
actly all the local minimum but instead one should rely
on other methods. For example (but not only) one could
implement algorithm developed in the realm of spin glass

![Fig. 1. Histogram of the local risk minima. The number of assets is $N = 16$ and the fixed portfolio return is the Nas-
daq return $R = R_{NAS} = 0.0014$. PDF stands for Probability Density Function.](image-url)
theory, aimed to freeze spin glass systems down to zero
temperature in order to reach the real minimum (ground
state). This rich class of algorithms can be roughly speak-
ing addressed as *simulated annealing* techniques [12,17].
If one try to implement this algorithms then it would be
almost impossible to end up on the efficient frontier, but
instead one would almost surely converge to a portfolio so-
lution corresponding to the average risk (see also section
4.4).

Moreover we observe that the proportions $\hat{p}$ can be
very different among themselves and from $\hat{p}_{MIN}$. This
is shown in Table (4) where are reported the portfolio $p_{MIN}$
(first column) corresponding to $\hat{\sigma}_{MIN}$, the portfolio $\hat{p}_{MAX}$
(last column) corresponding to $\hat{\sigma}_{MAX}$ and some portfo-
lios $\hat{p}$ (middle columns) having a risk value around $\hat{\sigma}_{AVE}$.
From a practical point of view, this *quasi-degeneracy of
the risk value with corresponding proportions $\hat{p}$ very far from each other* is the most interesting finding of this
paper. In order to illustrate better this point, we use a
very important and not trivial object in spin glass the-
tory, the *overlap distribution*. Let us denote the vector sign
of the global minimum solution $\bar{s} = \{\bar{s}_1, \ldots, \bar{s}_N\}$ and by
$s = \{\hat{s}_1, \ldots, \hat{s}_N\}$ another generic local minimum
solution (for simplicity, we consider only the sign of the propor-
tions $s_i = sign(p_i)$, but the same reasoning can be ex-
tended to the proportions themselves). A simple number,
which describe how different this two solutions are, can be
defined basically by counting how many $s_i$'s one must flip
in order to go from one configuration to the other. This
information is contained in the integer number $m(s, s)$ de-
fined as:

$$m(s, \hat{s}) = \frac{1}{2} \left( N - \sum_{k=1}^{N} \bar{s}_k \hat{s}_k \right)$$

and it ranges from 0 (identical sign vectors) to $N$ (opposite
sign vectors).

In the top side of Fig.(2) we plot the probability distribu-
tion function of the numbers $m_j$'s, obtained by measuring
the number of different signs between any single local min-
imum and the global solution: $m_j = m(\bar{s}, \bar{s}^{(j)})$, where $\bar{s}^{(j)}$
runs over all possible 6675 solutions. In particular, this
histogram show clearly that *most of* the solutions are in
turn very different from the one we would like to calculate
a priori, i.e. $\bar{s}$.

Actually, a little bit more than this can be said. In gen-
eral two different local solution might have almost the
same risk level but in a strategic-economic context they
can be totally different. This is shown in the bottom side
of Fig.(2), where the histogram of the overlaps between
all possible solutions is shown. Summarizing, a multiple
choice is available to the investor and an irreducible com-
ponent of arbitrariness is present in the final decision.
Moreover, it is likely that traditional method for recon-
strucnting the minimizing solution will lead the investor to be “trapped” into a quite different local minimum.

4.3 Exponential growth of solutions in the number of
assets

The impossibility of taking a rational decision described in
the previous paragraph get even worst if we consider the
dependence of the number of solutions from the number
of assets. We performed numerical experiments varying $N$
been found in reference [13] for the case of 20 assets of the New York Stock Exchange with no fixed portfolio return.

**Fig. 3.** The number of local risk minima versus the number of assets $N$. The fixed average return is the Nasdaq return $R_{NAS} = 0.0014$. The solid line is the best numerical fit $n(N) \sim \exp(0.69N)$.

This exponential growth of the number of risk local minima with the number of assets is the exact analog of the exponential growth of the number of local energy minima typical of spin glasses [19, 20]. From the economic point of view, the consequence of this growth is that the arbitrariness degree enlightened above get even bigger by increasing the portfolio dimension. Moreover we have verified (we do not report data for the sake of space) that the multiple solutions have a "chaoticity" property, in the sense that a small change of the correlation matrix $C$, or the addition of an extra asset, completely changes the values of optimal proportions. On the other hand the number of possible decisions decreases for increasing value of the fixed return $R$ under which minimization is performed.

Fig. 2. **Top:** the distribution of the overlap between the configuration $\tilde{s}$ corresponding to the risk global minimum and the others configurations $\tilde{s}^{(j)}$ corresponding to the risk local minima. **Bottom:** the distribution of the overlap between all the configuration $\tilde{s}^{(j)}$ (including $\tilde{s}$) corresponding to the risk local minima.

from 5 to 16 keeping the average return $R$ fixed to $R_{NAS}$. For each value we calculate the number of local risk minima $n(N)$, i.e. the number of solutions of Eq. (17). As it is clear from Fig. (3) this number grows exponentially with the number of assets (note the lin-log scale in the graph). The best numerical fit yields $n(N) \sim \exp(0.69N)$. We note that essentially the same value $n_{NY}(N) \sim \exp(0.68N)$ has
We argue this fixing $N = 16$ and calculating the number of risk local minima varying the return $R$ in the range $[0, 0.003]$. In Fig. (4) we plot the number of solutions of Eq. (17) corresponding to each $R$ value. We see that the local risk minimum decreases for increasing value of the return, becoming zero at $R \sim 0.0026$. Above these threshold we do not find from Eq. (17) any portfolios satisfying both the budget and the return constrains. One should look for risk minima on the border of the manifold where Lagrange optimization is performed. We did not investigate this point further.

![Graph showing the number of local risk minima versus the average portfolio return.](image)

**Fig. 4.** The number of local risk minima versus the average portfolio return. The number of assets is fixed to $N = 16$.

### 4.4 The efficient frontier

Here we use the data concerning the 16-stocks in order to compute the efficient frontier and two more related curves. More precisely, we first allow the averaged return to range from 0 to 0.003 in 100 constant steps and for each value of the return we calculated the proportions $p$ corresponding to the risk local minima. Then, among them we selected the one associated to the risk global minimum and the one associated to the worst choice, namely the local minimum corresponding to the portfolio at the very right tail of Fig. (1). Moreover, for a given fixed return, we also calculate the averaged risk $\sigma_{AVE}$. Namely, if $n$ is the total number of local minima for a fixed return $R$ and $\sigma^{(j)}$ are the associated risks ($j = 1, \ldots, n$), we define

$$\sigma_{AVE} = \frac{1}{n} \sum_{j=1}^{n} \sigma^{(j)}$$

By varying the return $R$, we use this data to reproduce a kind of averaged efficient frontier, which is shown in Fig. (5), together with the other two. We checked that, as one could expect, the proportion of the investment associated to the smallest and greatest local minima are completely different. Furthermore also the portfolios corresponding to risk value around the average risk $\sigma_{AVE}$ are very different, yielding many different equivalent investment strategies.

### 5 A few remarks on the variances/covariances matrix

A few words of comment about the correlation matrix are in place. The computation of matrix $C$ of variances/covariances between return variations of different assets is a key issue in the theoretical solution of the portfolio optimization problem. The study of these correlation matrices is a crucial problem in financial theory, both for developments of Markowitz's theory of optimal portfolios, and for problems (more recently introduced) of risk management, related to the so-called value at risk models. For covariance matrices used in value at risk models, the discussion is open.
Fig. 5. Efficient frontiers for the portfolio consisting of 16 assets. The continuous line is the “best-efficient” frontier corresponding to the lowest value between all the risk local minima. The dotted line shows the “average-efficient” frontier, i.e. it correspond to the average value of the risk between all the risk local minima. The dashed line is the “worst-efficient” frontier, constructed by using the higher values of the risk local minima for each fixed return.

To an almost continuous flow of new experiments, with an empirical evaluation of the obtained results, based on a single case (for instance see [22]).

In ref. [13] a random matrix approach is proposed by Galluccio et al. as an alternative to well-established index models -originally presented by Markowitz himself and developed by Sharpe [5]. These authors assume the matrix $C$ of variance/covariance as random; in particular, from an historical analysis on asset prices variations in various Stock Markets, they argue that the correlation matrix can be well approximated by a matrix which is a generic realization of the so-called Exponential Orthogonal Ensemble. In this way they can exploit a consolidated approach based on random matrix theory [23] and use self-averaging property to prove analytically the exponential increase of risk local minima. Assuming the matrix $C$ as random is quite reasonable in terms of financial theory: prices/returns movements are in any case random, since they can be read as realization of a stochastic process, generated by market fluctuations. The analysis as successively been enforced in a serie of papers [24–27] by examining the eigenvalues and eigenvectors distribution of the matrix itself. The main aim here is, to separate the randomness contained in the data from the real market information.

Let us consider a portfolio of $N$ assets: the correlation matrix contains $N(N-1)/2$ entries, which must be computed from $N$ time-series of length $T$. If $T$ is small compared to $N$, one would expect that the determination of the covariance is most likely to be noisy, and therefore the empirical correlation matrix is to a large extent random; this implies that the structure of the matrix is dominated by measurement noise and real information are somewhat hidden in the data. A deep analysis of eigenvalues (and corresponding eigenvectors) performed in ref [24,25] sheds relevant light on the statistical properties of empirical correlation matrices. In the case of the S & P500, less than 6% of the eigenvectors, which are responsible for 26% of the total volatility, appear to carry some information, and this is a surprising result. From this point of view, it should be stressed that Markovitz’s portfolio scheme, based on a purely historical determination of the correlation matrix, proves particularly weak, since the elements of the matrix itself are dominated by noise. Notwithstanding, simulations experiments with random matrices [28] show that, in the context of the classical portfolio problem (minimiz-
ing the portfolio variance under linear constraints) noise has relatively little effect. To leading order the solutions are determined by the stable, large eigenvalues, and the displacement of the solution due to noise is rather small. The picture is completely different, however, if we attempt to minimize the variance under non-linear constraint, like those we have in the problem of short-selling with margin account. In this problem the presence of noise in the correlation matrix leads to serious instability and a high degree of degeneracy of the solutions.

6 Conclusions

In this paper we have presented a model of portfolio optimization in the general case of futures markets. That is allowing for long buying/short selling of assets with a fixed margin requirement, and assuming a relatively high number of assets. In this perspective, this model generalizes some relevant results originally obtained by Lintner. The introduction of a nonlinear constraint in the Lagrangian function make the optimization procedure to find the solution very difficult. Firstly, it is not possible to find a unique efficiency frontier because of the presence in the solution equation of a vector composed by a sequence of ±1 (corresponding to buying or selling the single asset), whose order is undefined. The consequence is that we have a multiple equilibrium solution characterized by many local minima. The number of these minima is exponentially increasing function in the number of assets, like the number of metastable states in spin glasses. We have illustrated the analogies between the two models. In front of the problem of having many efficiency frontiers, all corresponding to a (local) minimum risk but not equivalent among them, we have implemented a successful numerical procedure to search for the minimum of all minima, in such a way to find a sort of super-efficiency frontier, that will allow to get the best portfolio in terms of relationship between risk and return. We have applied our model to a concrete portfolio, formed by 16 assets chosen among the most traded ones at Nasdaq. We went through the whole two-step procedure, obtaining significant results, which may suggest further efforts to develop the presented model.

Acknowledgements

The authors would like to thank J.-P. Bouchaud, Raffaella Burioni, Sandro Graffi, Mario Menegatti and Paola Modesti for helpfull comments and suggestions.

References

2. B. Mandelbrot, J. Business 36, 394 (1963)


<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Adobe</td>
<td>ADO</td>
<td>0.004818</td>
</tr>
<tr>
<td>2 Amazon</td>
<td>AMA</td>
<td>0.002838</td>
</tr>
<tr>
<td>3 Ameritrade</td>
<td>AME</td>
<td>0.005522</td>
</tr>
<tr>
<td>4 Aol</td>
<td>AOL</td>
<td>0.003374</td>
</tr>
<tr>
<td>5 Appmc</td>
<td>APP</td>
<td>0.008313</td>
</tr>
<tr>
<td>6 Broadvis</td>
<td>BRO</td>
<td>0.008397</td>
</tr>
<tr>
<td>7 Cisco</td>
<td>CIS</td>
<td>0.002844</td>
</tr>
<tr>
<td>8 Cmg</td>
<td>CMG</td>
<td>0.004129</td>
</tr>
<tr>
<td>9 Dell</td>
<td>DEL</td>
<td>0.000263</td>
</tr>
<tr>
<td>10 Doubleclick</td>
<td>DOU</td>
<td>0.004788</td>
</tr>
<tr>
<td>11 Ebay</td>
<td>EBA</td>
<td>0.005765</td>
</tr>
<tr>
<td>12 Inktomy</td>
<td>INK</td>
<td>0.004144</td>
</tr>
<tr>
<td>13 Intel</td>
<td>INT</td>
<td>0.001683</td>
</tr>
<tr>
<td>14 Jdsumi</td>
<td>JDS</td>
<td>0.005999</td>
</tr>
<tr>
<td>15 Microsof</td>
<td>MIC</td>
<td>0.000843</td>
</tr>
<tr>
<td>16 Oracle</td>
<td>ORA</td>
<td>0.004045</td>
</tr>
<tr>
<td>** Nasdaq Index</td>
<td>NAS</td>
<td>0.001362</td>
</tr>
</tbody>
</table>

*Table 1. Rate of Return*
<table>
<thead>
<tr>
<th></th>
<th>ADO</th>
<th>AMA</th>
<th>AME</th>
<th>AOL</th>
<th>APP</th>
<th>BRO</th>
<th>CIS</th>
<th>CMG</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADO</td>
<td>0.0017655</td>
<td>0.000678</td>
<td>0.0005875</td>
<td>0.0004569</td>
<td>0.0010777</td>
<td>0.001906</td>
<td>0.000633</td>
<td>0.0009492</td>
</tr>
<tr>
<td>AMA</td>
<td>0.006738</td>
<td>0.0039831</td>
<td>0.0015211</td>
<td>0.0014186</td>
<td>0.000982</td>
<td>0.0013937</td>
<td>0.0008632</td>
<td>0.0025806</td>
</tr>
<tr>
<td>AME</td>
<td>0.0005875</td>
<td>0.0015211</td>
<td>0.005604</td>
<td>0.0014733</td>
<td>0.000982</td>
<td>0.0013879</td>
<td>0.0007159</td>
<td>0.0021351</td>
</tr>
<tr>
<td>AOL</td>
<td>0.0004569</td>
<td>0.0014186</td>
<td>0.014733</td>
<td>0.0018082</td>
<td>0.0006482</td>
<td>0.0010780</td>
<td>0.0006245</td>
<td>0.0017036</td>
</tr>
<tr>
<td>APP</td>
<td>0.0010777</td>
<td>0.0009082</td>
<td>0.0006482</td>
<td>0.0038179</td>
<td>0.0016104</td>
<td>0.0011169</td>
<td>0.0017814</td>
<td>0.0011875</td>
</tr>
<tr>
<td>BRO</td>
<td>0.0007096</td>
<td>0.0013937</td>
<td>0.0013879</td>
<td>0.0010780</td>
<td>0.0016104</td>
<td>0.0032910</td>
<td>0.0008336</td>
<td>0.0020747</td>
</tr>
<tr>
<td>CIS</td>
<td>0.000633</td>
<td>0.0008632</td>
<td>0.0007159</td>
<td>0.0006245</td>
<td>0.0011169</td>
<td>0.0008336</td>
<td>0.001123</td>
<td>0.0011875</td>
</tr>
<tr>
<td>CMG</td>
<td>0.0009492</td>
<td>0.0025806</td>
<td>0.0012351</td>
<td>0.0017036</td>
<td>0.0017814</td>
<td>0.0020747</td>
<td>0.00185</td>
<td>0.001836</td>
</tr>
<tr>
<td>DEL</td>
<td>0.0004815</td>
<td>0.0006627</td>
<td>0.0006939</td>
<td>0.0006952</td>
<td>0.0006867</td>
<td>0.0007166</td>
<td>0.0011147</td>
<td>0.0025681</td>
</tr>
<tr>
<td>DOU</td>
<td>0.0006174</td>
<td>0.0019525</td>
<td>0.0022906</td>
<td>0.0013104</td>
<td>0.0012851</td>
<td>0.0017050</td>
<td>0.0009150</td>
<td>0.0025681</td>
</tr>
<tr>
<td>EBA</td>
<td>0.0007485</td>
<td>0.0022906</td>
<td>0.0016129</td>
<td>0.0013104</td>
<td>0.0012851</td>
<td>0.0017050</td>
<td>0.0009150</td>
<td>0.0025681</td>
</tr>
<tr>
<td>INK</td>
<td>0.0006859</td>
<td>0.0019766</td>
<td>0.0015506</td>
<td>0.0012455</td>
<td>0.0013270</td>
<td>0.0020950</td>
<td>0.0009573</td>
<td>0.0026140</td>
</tr>
<tr>
<td>INT</td>
<td>0.0005465</td>
<td>0.0006466</td>
<td>0.0005004</td>
<td>0.0005156</td>
<td>0.0009470</td>
<td>0.0006798</td>
<td>0.0006638</td>
<td>0.0009792</td>
</tr>
<tr>
<td>JDS</td>
<td>0.0007854</td>
<td>0.0008836</td>
<td>0.0009315</td>
<td>0.0007894</td>
<td>0.0016879</td>
<td>0.0013783</td>
<td>0.0010121</td>
<td>0.0015445</td>
</tr>
<tr>
<td>MIC</td>
<td>0.0003891</td>
<td>0.0005764</td>
<td>0.0004497</td>
<td>0.000204</td>
<td>0.0005583</td>
<td>0.0006681</td>
<td>0.0004818</td>
<td>0.0007499</td>
</tr>
<tr>
<td>ORA</td>
<td>0.0007064</td>
<td>0.0008636</td>
<td>0.0007249</td>
<td>0.000589</td>
<td>0.0011963</td>
<td>0.0009879</td>
<td>0.0007990</td>
<td>0.0012780</td>
</tr>
</tbody>
</table>

Table 2. Variance/Covariance matrix

<table>
<thead>
<tr>
<th></th>
<th>DEL</th>
<th>DOU</th>
<th>EBA</th>
<th>INK</th>
<th>INT</th>
<th>JDS</th>
<th>MIC</th>
<th>ORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADO</td>
<td>0.0004815</td>
<td>0.0006174</td>
<td>0.0007485</td>
<td>0.0005659</td>
<td>0.0007854</td>
<td>0.0003891</td>
<td>0.0007064</td>
<td></td>
</tr>
<tr>
<td>AMA</td>
<td>0.0006627</td>
<td>0.0019525</td>
<td>0.0022906</td>
<td>0.0019766</td>
<td>0.0006466</td>
<td>0.0008362</td>
<td>0.0008636</td>
<td></td>
</tr>
<tr>
<td>AME</td>
<td>0.000693</td>
<td>0.0022906</td>
<td>0.0016129</td>
<td>0.0015506</td>
<td>0.0005004</td>
<td>0.0009315</td>
<td>0.0009470</td>
<td></td>
</tr>
<tr>
<td>AOL</td>
<td>0.0006179</td>
<td>0.0014429</td>
<td>0.0013104</td>
<td>0.0012455</td>
<td>0.0005156</td>
<td>0.0007894</td>
<td>0.000424</td>
<td></td>
</tr>
<tr>
<td>APP</td>
<td>0.0006952</td>
<td>0.0013270</td>
<td>0.0012851</td>
<td>0.0013270</td>
<td>0.0009470</td>
<td>0.0016879</td>
<td>0.0005583</td>
<td></td>
</tr>
<tr>
<td>BRO</td>
<td>0.0006867</td>
<td>0.0020950</td>
<td>0.0017050</td>
<td>0.0020950</td>
<td>0.000698</td>
<td>0.0013783</td>
<td>0.0006681</td>
<td></td>
</tr>
<tr>
<td>CIS</td>
<td>0.0007166</td>
<td>0.0009573</td>
<td>0.0009150</td>
<td>0.0009573</td>
<td>0.0006638</td>
<td>0.0010121</td>
<td>0.0009790</td>
<td></td>
</tr>
<tr>
<td>CMG</td>
<td>0.0011147</td>
<td>0.0026140</td>
<td>0.0025681</td>
<td>0.0026140</td>
<td>0.0009792</td>
<td>0.0015445</td>
<td>0.0007499</td>
<td></td>
</tr>
<tr>
<td>DEL</td>
<td>0.0014036</td>
<td>0.0007637</td>
<td>0.0008658</td>
<td>0.0008183</td>
<td>0.0007496</td>
<td>0.0006910</td>
<td>0.0005324</td>
<td></td>
</tr>
<tr>
<td>DOU</td>
<td>0.0007637</td>
<td>0.0005032</td>
<td>0.0023282</td>
<td>0.004071</td>
<td>0.0007030</td>
<td>0.0012190</td>
<td>0.0005463</td>
<td></td>
</tr>
<tr>
<td>EBA</td>
<td>0.0008658</td>
<td>0.0018463</td>
<td>0.0020844</td>
<td>0.0020844</td>
<td>0.000491</td>
<td>0.0006221</td>
<td>0.0006542</td>
<td></td>
</tr>
<tr>
<td>INK</td>
<td>0.0008183</td>
<td>0.0020844</td>
<td>0.004071</td>
<td>0.004071</td>
<td>0.000491</td>
<td>0.0006221</td>
<td>0.0006542</td>
<td></td>
</tr>
<tr>
<td>INT</td>
<td>0.0007496</td>
<td>0.0006473</td>
<td>0.000689</td>
<td>0.0007030</td>
<td>0.0011612</td>
<td>0.0008158</td>
<td>0.0004691</td>
<td></td>
</tr>
<tr>
<td>JDS</td>
<td>0.0006910</td>
<td>0.0009491</td>
<td>0.0010614</td>
<td>0.0012190</td>
<td>0.0008158</td>
<td>0.0025830</td>
<td>0.0005088</td>
<td></td>
</tr>
<tr>
<td>MIC</td>
<td>0.0005324</td>
<td>0.0006221</td>
<td>0.0005463</td>
<td>0.0005463</td>
<td>0.000491</td>
<td>0.0006221</td>
<td>0.0006542</td>
<td></td>
</tr>
<tr>
<td>ORA</td>
<td>0.0005532</td>
<td>0.0006542</td>
<td>0.0008820</td>
<td>0.000930</td>
<td>0.0005454</td>
<td>0.0010459</td>
<td>0.0004617</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Variance/Covariance matrix (Continued)
<table>
<thead>
<tr>
<th>Risk</th>
<th>0.0090</th>
<th>0.0100</th>
<th>0.0100</th>
<th>0.0100</th>
<th>0.0100</th>
<th>0.0100</th>
<th>0.0100</th>
<th>0.0162</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADO</td>
<td>0.0630</td>
<td>-0.0070</td>
<td>0.0780</td>
<td>0.0890</td>
<td>0.0890</td>
<td>0.1070</td>
<td>0.0990</td>
<td>0.1100</td>
</tr>
<tr>
<td>AMA</td>
<td>-0.0460</td>
<td>-0.0520</td>
<td>-0.0390</td>
<td>-0.0260</td>
<td>-0.0320</td>
<td>-0.0380</td>
<td>-0.0080</td>
<td>0.0250</td>
</tr>
<tr>
<td>AME</td>
<td>0.0150</td>
<td>0.0210</td>
<td>0.0330</td>
<td>-0.0020</td>
<td>-0.0160</td>
<td>0.0170</td>
<td>0.0330</td>
<td>0.0310</td>
</tr>
<tr>
<td>AOL</td>
<td>0.0820</td>
<td>0.0950</td>
<td>-0.0280</td>
<td>-0.0110</td>
<td>0.1480</td>
<td>0.0900</td>
<td>-0.0180</td>
<td>0.1250</td>
</tr>
<tr>
<td>APP</td>
<td>0.0320</td>
<td>0.0600</td>
<td>0.0380</td>
<td>0.0480</td>
<td>0.0660</td>
<td>0.0800</td>
<td>0.0720</td>
<td>0.0280</td>
</tr>
<tr>
<td>BRO</td>
<td>0.0390</td>
<td>0.0340</td>
<td>0.0340</td>
<td>0.0260</td>
<td>0.0360</td>
<td>0.0350</td>
<td>0.0410</td>
<td>0.0550</td>
</tr>
<tr>
<td>CIS</td>
<td>0.1230</td>
<td>-0.1610</td>
<td>-0.1770</td>
<td>-0.1570</td>
<td>-0.1280</td>
<td>-0.1720</td>
<td>-0.1460</td>
<td>0.0670</td>
</tr>
<tr>
<td>CMG</td>
<td>-0.0450</td>
<td>0.0030</td>
<td>0.0170</td>
<td>-0.0390</td>
<td>-0.0330</td>
<td>0.0020</td>
<td>-0.0440</td>
<td>0.0120</td>
</tr>
<tr>
<td>DEL</td>
<td>-0.1380</td>
<td>-0.1600</td>
<td>-0.1520</td>
<td>-0.0800</td>
<td>-0.0390</td>
<td>0.0410</td>
<td>0.1070</td>
<td>0.0000</td>
</tr>
<tr>
<td>DOU</td>
<td>0.0220</td>
<td>-0.0230</td>
<td>-0.0170</td>
<td>0.0320</td>
<td>-0.0150</td>
<td>-0.0280</td>
<td>-0.0100</td>
<td>-0.1100</td>
</tr>
<tr>
<td>EBA</td>
<td>0.0410</td>
<td>0.0490</td>
<td>0.0520</td>
<td>0.0510</td>
<td>0.0460</td>
<td>0.0320</td>
<td>0.0340</td>
<td>0.0420</td>
</tr>
<tr>
<td>INK</td>
<td>-0.0300</td>
<td>-0.0320</td>
<td>-0.0290</td>
<td>-0.0250</td>
<td>-0.0280</td>
<td>-0.0300</td>
<td>0.0180</td>
<td>-0.1640</td>
</tr>
<tr>
<td>INT</td>
<td>0.0890</td>
<td>0.0970</td>
<td>0.1010</td>
<td>-0.1090</td>
<td>-0.0760</td>
<td>-0.1470</td>
<td>-0.1140</td>
<td>0.0560</td>
</tr>
<tr>
<td>JDS</td>
<td>-0.0440</td>
<td>0.0580</td>
<td>0.0630</td>
<td>0.0870</td>
<td>0.0860</td>
<td>0.0820</td>
<td>0.0930</td>
<td>0.0100</td>
</tr>
<tr>
<td>MIC</td>
<td>-0.1490</td>
<td>0.0900</td>
<td>0.0950</td>
<td>0.1400</td>
<td>-0.1080</td>
<td>0.0800</td>
<td>-0.1360</td>
<td>0.1720</td>
</tr>
<tr>
<td>ORA</td>
<td>0.0420</td>
<td>0.0570</td>
<td>0.0470</td>
<td>0.0760</td>
<td>-0.0370</td>
<td>-0.0260</td>
<td>-0.0170</td>
<td>0.0300</td>
</tr>
</tbody>
</table>

Table 4. Some portfolios corresponding to fixed return $R = R_{NAS} = 0.0014$. The first column is the “best” portfolio corresponding to the global risk minimum $\hat{\sigma}_{MIN} \approx 0.0090$. The last column is the “worst” portfolio corresponding to the highest of the local risk minima $\hat{\sigma}_{MAX} \approx 0.0162$. In the middle columns there are some portfolios having risk around the average risk value $\hat{\sigma}_{AVE} \approx 0.0100$.