Exact formulas for two interacting particles and applications in particle systems with duality

Gioia Carinci[a]
Cristian Giardinà[b]
Frank Redig[a]

[a] Delft Institute of Applied Mathematics, Technische Universiteit Delft
Mekelweg 4, 2628 CD Delft, The Netherlands
[b] Department of Mathematics, University of Modena and Reggio Emilia
via G. Campi 213/b, 41125 Modena, Italy

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Abstract

We consider two particles performing continuous-time nearest neighbor random walk on $\mathbb{Z}$ and interacting with each other when they are at neighboring positions. Typical examples are two particles in the partial exclusion process or in the inclusion process. We provide an *exact formula* for the Laplace-Fourier transform of the transition probabilities of the two-particle dynamics. From this we derive a general *scaling limit* result, which shows that the possible scaling limits are coalescing Brownian motions, reflected Brownian motions, and sticky Brownian motions.

In particle systems with duality, the solution of the dynamics of two dual particles provides relevant information. We apply the exact formula to the the symmetric inclusion process, that is self-dual, in the *condensation regime*. We thus obtain two results. First, by computing the time-dependent covariance of the particle occupation number at two lattice sites we characterise the time-dependent coarsening in infinite volume when the process is started from a homogeneous product measure. Second, we identify the limiting variance of the density field in the diffusive scaling limit, relating it to the local time of sticky Brownian motion.

1 Introduction

In interacting particle systems, duality is a powerful tool which enables to study time-dependent *correlation functions of order* n with the help of n *dual particles*. Examples include the exclusion processes [8, 16] and the inclusion processes [10], related diffusion processes such as the Brownian Energy process [9], and stochastic energy exchange processes, such as Kipnis-Marchioro-Presutti model [14].

For systems defined on the infinite lattice $\mathbb{Z}^d$, these dual particles typically spread out and behave on large scale as independent Brownian motions. This fact is usually enough to prove the hydrodynamic limit in the sense of propagation of local equilibrium [8]: under a diffusive scaling limit the macroscopic density profile evolves according to the heat equation. To study the *fluctuations of the density field* one needs to understand *two dual particles*. Likewise, solving the dynamics of a finite number n of dual particles one gets control on the n^{th} moment of the density field. While the dynamics of one dual particle is usually easy to deal with (being typically a continuous-time random walk), the dynamics of n dual particles is usually an hard problem (due to the interaction of the dual walkers) that can be solved only in special systems, i.e. stochastic integrable systems [3, 4, 17, 19, 22, 23]. In such systems the equations for the time-dependent correlation functions essentially decouple and the transition probability of a finite number of dual particles can be solved using methods related to quantum integrability, e.g. Bethe ansatz,Yang Baxter equation, and factorized S-matrix. For instance, by studying the two-particle problem for the asymmetric simple exclusion process, it was proved in [20] that the probability distribution of the particle density of only two particles spreads in time diffusively, but with a diffusion coefficient that is notably different from the non-interacting case.

In this paper we prove that in dimension $d = 1$ and for nearest neighbor jumps with translation invariant rates, a generic system of two (dual) particles turns out to be *exactly solvable*, i.e., one can obtain a closed-form formula for the Laplace-Fourier transform of the transition probabilities in the coordinates of the center of mass and the distance between the particles. The derivation of this formula, and its applications in particle systems with duality, is the main message of this paper.

The exact formula for the two-particle process will be stated in Theorem 2.8. We shall provide conditions on the particles jump rates so that the formula holds. These conditions include both the case where the particles have symmetric interactions or the case where the asymmetry is “naive”, meaning that it is obtained from the symmetric system by multiplying the rates of jumps to the right by a parameter p and the rates of jumps to the left by a parameter q, with $p \neq q$.

From Theorem 2.8 we obtain a general *scaling limit* result (Theorem 2.11), which shows that the possible scaling limits of two interacting random walkers are coalescing Brownian motions (“absorbed regime”), reflecting Brownian motions (“reflected regime”) and *sticky Brownian motions* (attracting each other via their local intersection time) which interpolate between the absorbed and reflected regime, and where the particles spend some positive proportion of time at the same place.

Next, we consider applications of Theorem 2.8 to systems with duality. We focus, in particular, on the inclusion process [10]. Due to the attractive interaction between particles this model has a condensation regime [11, 1]. As a consequence of Theorem 2.11, the scaling limit of the two-particle dynamics yields sticky Brownian motions. Furthermore, by using duality, we study the scaling behavior of the variance and covariance of the number of particles in the condensation regime (Theorem 2.16). This provides better understanding of the coarsening process (building up of large piles of particles) when starting from a homogeneous initial product measure in infinite volume. Last, we study in Theorem 2.18 the time-dependent variance of the density fluctuation field of the inclusion process in the condensation regime. We conjecture the expression that we find in Theorem 2.18 to be a universal expression describing the time-dependent variance of the density fluctuation field of systems displaying condensation phenomena, when started from a non-equilibrium initial state.

2 Model definitions and results

2.1 The setting

We start by defining a general system of particles moving on $\mathbb{Z}$ and interacting when they are nearest neighbor. The system is modeled by a continuos-time Markov chain and thus is defined by assigning the process generator.

Definition 2.1 (Generator)**.** *Let $\{\eta(t) : t \geq 0\}$ be a particle system on the integer lattice, where $\eta_i(t) \in \Omega \subseteq \mathbb{N}$ denotes the number of particles at site $i \in \mathbb{Z}$ at time t. The particles evolve according to the formal generator:*

$$[\mathscr{L} f](\eta) = \sum_{i\in\mathbb{Z}} \left\{ c_+(\eta_i, \eta_{i+1}) \left[f(\eta^{i,i+1}) - f(\eta) \right] + c_-(\eta_{i+1}, \eta_i) \left[f(\eta^{i+1,i}) - f(\eta) \right] \right\} . \quad (1)$$

In the above, $\eta^{i,j}$ is the configuration that is obtained from $\eta \in \Omega^{\mathbb{Z}}$ by removing a particle at site i and adding it at site j, i.e.

$$\eta_k^{i,j} = \begin{cases} \eta_i - 1 & \text{if } k = i, \\ \eta_j + 1 & \text{if } k = j, \\ \eta_k & \text{otherwise} . \end{cases}$$

The specific systems that we shall consider below include cases where η_i, the number of particles at site i, has a maximum (i.e. Ω is a finite set, as in the partial exclusion processes [17]), as well as cases where any natural number is allowed (i.e. $\Omega = \mathbb{N}$, as in the inclusion processes [10]). The rates $c_+(\eta_i, \eta_{i+1})$ and $c_-(\eta_{i+1}, \eta_i)$ in (1) are allowed to be general functions of the particle numbers η_i and η_{i+1} such that there is a well defined Markov process associated to the formal generator $\mathscr{L}$.

When the rates of the right jumps are equal to the rates of the left jumps, i.e.

$$c_+(n, m) = c_-(n, m) \qquad \forall \ n, m \in \Omega,$$

we shall say that the system is *symmetric*. For some of our results, symmetry of the jumps is not required. However, if there is asymmetry, we need it to be of a specific type that is explained below.

2.2 Two interacting particles

We shall be interested in the case where the process $\{\eta(t) : t \geq 0\}$ with generator (1) is initialized with *two particles*. In the following we will restrict to the case where the following conditions are fulfilled.

Condition 2.2 (Rates of two-particle process)**.** *For the rates in* (1) *we assume that:*

(i) they are translation invariant, i.e. for all $\eta \in \Omega^{\mathbb{Z}}$ and for all $a \in \mathbb{Z}$

$$\begin{aligned} c_+(\eta_i, \eta_{i+1}) &= c_+((\tau_a \eta)_{i+a}, (\tau_a \eta)_{i+1+a}) \\ c_-(\eta_{i+1}, \eta_i) &= c_-((\tau_a \eta)_{i+1+a}, (\tau_a \eta)_{i+a}), \end{aligned}$$

where τ_a denotes the shift by a on $\mathbb{Z}$, i.e. $(\tau_a \eta)_i = \eta_{i-a}$;

(ii) for integers couples (n, m) such that $n + m \leq 2$, they satisfy

$$c_+(2,0) + c_-(2,0) = 2\,(c_+(1,0) + c_-(1,0))\ , \tag{2}$$

$$\frac{c_+(1,0)}{c_-(1,0)} = \frac{c_+(1,1)}{c_-(1,1)} = \frac{c_+(2,0)}{c_-(2,0)}\ . \tag{3}$$

Equation (2) is for instance implied by linearity of the rates in the number of particles at the departure site, i.e. $c_+(\eta_i, 0) = A\eta_i$ and $c_-(\eta_{i+1}, 0) = B\eta_{i+1}$ with $A, B > 0$. Equation (3) refers instead to the degree of asymmetry in the jumps: it is for instance satisfied when the ratio

$$\frac{c_+(\eta_i, \eta_{i+1})}{c_-(\eta_i, \eta_{i+1})}$$

is a constant, i.e. does not depend on η_i, η_{i+1}. This happens in symmetric systems or in systems with *naive asymmetry* that are obtained from the symmetric systems multiplying by two different constants the rates of the left jumps and those of the right jumps.

Given rates that fulfill Condition 2.2, it will be convenient to define three parameters that characterize the dynamics of two particles.

Definition 2.3 (Parameters)**.** *We define the parameters*

$$\alpha := c_+(1,0) + c_-(1,0), \tag{4}$$

$$p := \frac{1}{\alpha}\, c_+(1,0) \tag{5}$$

$$\theta := \frac{c_+(1,1) + c_-(1,1)}{c_+(1,0) + c_-(1,0)} - 1\ . \tag{6}$$

Remark 2.4. *Clearly, to define in general the two-particle process one needs to assign six rates, namely $c_\pm(1,0)$, $c_\pm(2,0)$ and $c_\pm(1,1)$. If Condition 2.2 holds then equations* (2) *and* (3) *leave three free choices. The parameters defined above are to be interpreted as follows:*

1. *$\alpha > 0$ is total rate for a single particle to jump if both left and right neighboring sites are empty;*

2. $0 < p < 1$ *is the probability of a particle to jump to the right when its neighboring sites are empty; for convenience we denote by* $q = 1 - p$ *the probability of a left jump.*

3. $\theta \in \mathbb{R}$ *is a parameter tuning the strength of the interaction between the two particles. In particular, for* $\theta = 0$ *one recovers the case where the particles perform two independent random walks.*

2.3 Distance and center of mass coordinates

One main goal of this paper is to achieve a full control of the dynamics of the two-particle process. To this aim it will be convenient to move to new coordinates. Consider the process $\{\eta(t) : t \geq 0\}$ with generator (1) initialized with two particles and denote by $(x_1(t), x_2(t))$ the particle positions at time t, with an arbitrary labeling of the particles, but fixed once for all. Define the *distance* and *sum* coordinates by

$$\begin{aligned} w(t) &:= |x_2(t) - x_1(t)| \, , \\ u(t) &:= \ \ x_1(t) + x_2(t) \, . \end{aligned} \tag{7}$$

By definition, the distance and sum coordinates are not depending on the chosen labeling of particles. As a consequence of the fact that the size of the particle jumps is one, both the difference and the sum coordinates change by one unit at every particle jump. Therefore they both perform continuous-time simple random walk. Under Condition 2.2, a straightforward computation starting from (1) shows that the distance process $\{w(t) : t \geq 0\}$, that is valued in $\mathbb{N} \cup \{0\}$, evolves according to the generator

$$[\mathscr{L}^{(dist)} f](w) = \begin{cases} 2\alpha(f(w+1) - f(w)) & \text{if } w = 0, \\ \\ \alpha(f(w+1) - f(w)) + \alpha(\theta+1)(f(w-1) - f(w)) & \text{if } w = 1, \\ \\ \alpha(f(w+1) - 2f(w) + f(w-1)) & \text{if } w \geq 2. \end{cases} \tag{8}$$

As for the sum coordinate $\{u(t) \, : \, t \geq 0\}$, this is a process that is valued in $\mathbb{Z}$. One finds that, *conditionally on* $\{w(t), t \geq 0\}$, it evolves according to

$$[\mathscr{L}^{(sum)} f](u) = 2\alpha \left(1 + \frac{\theta}{2}\, \mathbf{1}_{w=1}\right) \{p[f(u+1) - f(u)] + q[f(u-1) - f(u)]\} \, . \tag{9}$$

We thus see that the distance between the particles evolves in an *autonomous* way as a symmetric random walk on the integers, reflected at 0 and with a defect in 1. The sum coordinate instead is dependent on the distance through its jump rate. More precisely, the sum performs an asymmetric continuous-time nearest neighbor random walk driven by a Poisson process whose rate depends on the distance process. These properties of the distance and sum coordinates will be the key properties that will be used in the exact solution of the two-particle dynamics.

Such exact solution will be expressed by considering the Fourier-Laplace transform of the transition probability

$$P_t((u,w),(u',w')) = \mathbb{P}(u(t) = u', w(t) = w' \,|\, u(0) = u, w(0) = w), \tag{10}$$

where $\mathbb{P}$ denotes the law of the two-particle process. As it can be seen from the generators (8) and (9), these transition probabilities are translation invariant only in the sum coordinate, i.e., $P_t((u,w),(u',w')) = P_t((0,w),(u'-u,w'))$, and therefore it is natural that we take Fourier transform w.r.t. the sum coordinate. Furthermore it will also be convenient to take Laplace transform w.r.t. time.

Definition 2.5 (Fourier-Laplace transform of the transition probability)**.** *Let the parameter α defined in* (4) *be equal to* 1 *and let $P_t\left((u,w),(u',w')\right)$ be the transition probability in* (10)*. We define the Laplace transform of the transition probability*

$$\mathscr{G}^{(\theta)}((u,w),(u',w');\lambda) := \int_0^\infty e^{-\lambda t} P_t\left((u,w),(u',w')\right)\, dt, \qquad \lambda \geq 0 \tag{11}$$

and its Fourier transform

$$G^{(\theta)}(w,w',\kappa,\lambda) := \sum_{v\in\mathbb{Z}} e^{-i\kappa v}\, \mathscr{G}^{(\theta)}\left((0,w),(v,w');\lambda\right), \qquad \kappa \in \mathbb{R}\,. \tag{12}$$

Remark 2.6 (Changing α)**.** *Notice that the parameter $\alpha > 0$ in* (9) *appears as a multiplicative factor in the generator, therefore for a generic value of this parameter we have the scaling property*

$$\mathscr{G}^{(\theta,\alpha)}((u,w),(u',w');\lambda) = \frac{1}{\alpha}\mathscr{G}^{(\theta)}\left((u,w),(u',w');\frac{\lambda}{\alpha}\right), \tag{13}$$

where we made the α-dependence explicit. The $\mathscr{G}^{(\theta)}$ in (11) *coincides with $\mathscr{G}^{(\theta,1)}$ and for a generic value of α we can use* (13)*.*

2.4 Processes with duality

Besides the inherent interest of studying the two particle process (and its scaling limits) we will be interested in applications of the exact solution of the two-particle dynamics. This applications will be given in the context of interacting particle systems with self-duality.

Definition 2.7 (Self-duality)**.** *Let $\{\eta(t) : t \geq 0\}$ be a process of type introduced in Definition 2.1. We say that the process is self-dual with self-duality function $D : \Omega \times \Omega \to \mathbb{R}$ if for all $t \geq 0$ and for all $\eta, \xi \in \Omega^{\mathbb{Z}}$ we have the self-duality relation*

$$\mathbb{E}_\eta D(\xi, \eta(t)) = \mathbb{E}_\xi D(\xi(t), \eta), \tag{14}$$

where $\{\xi(t) : t \geq 0\}$ is an independent copy of the process with generator (1)*. In the above $\mathbb{E}_\eta$ on the l.h.s. denotes expectation in the original process initialized from the configuration η and $\mathbb{E}_\xi$ on the r.h.s. denotes expectation in the copy process initialized from the configuration ξ. We shall call D a factorized self-duality function when*

$$D(\xi,\eta) = \prod_{i\in\mathbb{Z}} d(\xi_i, \eta_i). \tag{15}$$

The function $d(\cdot,\cdot)$ is then called the single-site self-duality function.

For self-dual processes the dynamics of two particles provides relevant information about the time-dependent correlation functions of degree two. The simplest possible choice satisfying Conditions 2.2 and the self-duality property is the choice of rates that are linear in both the departure and arrival sites. For later convenience we shall refer to the process with this choice of the rates as the *reference process*. It is given by the generator

$$[\mathscr{L}_{\rm ref} f](\eta) = \frac{\alpha}{2} \sum_{i\in\mathbb{Z}} \left\{ \eta_i(1+\theta\eta_{i+1}) \left[f(\eta^{i,i+1}) - f(\eta)\right] + \eta_{i+1}(1+\theta\eta_i) \left[f(\eta^{i+1,i}) - f(\eta)\right] \right\} . \tag{16}$$

This is a process whose behavior depends on the sign of θ. It corresponds to the exclusion process for $\theta < 0$, to the inclusion process for $\theta > 0$, and to independent random walkers for $\theta = 0$. We now explain the precise connection between these processes and the reference process (16).

2.4.1 The symmetric inclusion process

The Symmetric Inclusion Process with parameter $k > 0$, denoted SIP(k), is the process with generator [10]

$$L_{{\rm SIP}(k)} f(\eta) = \sum_{i\in\mathbb{Z}} \left(\eta_i(k+\eta_{i+1})\nabla_{i,i+1} + \eta_{i+1}(k+\eta_i)\nabla_{i+1,i}\right) f(\eta), \tag{17}$$

where $\nabla_{i,\ell} f(\eta) = f(\eta^{i,\ell}) - f(\eta)$. This amounts to choose the parameters in the reference process (16) as follows

$$\theta = \frac{1}{k} \qquad \text{and} \qquad \alpha = \frac{2}{\theta} = 2k, \tag{18}$$

Conversely, the reference process with generator (16) corresponds to a time rescaling of the SIP process, i.e.

$$\eta^{\rm ref}(t) = \eta^{{\rm SIP}(\frac{1}{\theta})}\left(\frac{\theta\alpha}{2} t\right). \tag{19}$$

The SIP(k) is self dual with single-site self-duality function:

$$d_{{\rm SIP}(k)}(m,n) = \frac{n!\Gamma(k)}{(n-m)!\Gamma(k+m)}\, \mathbf{1}_{\{m\le n\}}. \tag{20}$$

This self-duality property with self-duality function (20) continues to hold when $\theta = \frac{1}{k}$ for all other values of $\alpha > 0$.

2.4.2 The symmetric partial exclusion process

We recall the definition of the Symmetric partial Exclusion Process with parameter $j \in \mathbb{N}$, SEP(j) [21]. Notice that j, that is the maximum number of particles allowed for each site, has to be a natural number. For $j = 1$ the process is the standard Symmetric Exclusion Process. The generator is

$$L_{{\rm SEP}(j)} f(\eta) = \sum_{i\in\mathbb{Z}} \left(\eta_i(j-\eta_{i+1})\nabla_{i,i+1} + \eta_{i+1}(j-\eta_i)\nabla_{i+1,i}\right) f(\eta), \tag{21}$$

i.e., comparing with the reference process (16) we have

$$\theta = -\frac{1}{j}, \qquad \alpha = -\frac{2}{\theta} = 2j. \tag{22}$$

The Symmetric partial Exclusion Process SEP(j) is self-dual with single-site self-duality function:

$$d_{\mathrm{SEP}(j)}(m,n) = \frac{\binom{n}{m}}{\binom{j}{m}}\, \mathbf{1}_{\{m\le n\}}. \tag{23}$$

As before, this self-duality property with self-duality function (23) continues to hold when $\theta = -1/j$ for all other values of $\alpha > 0$.

2.4.3 Independent symmetric random walk

The last example is provided by a system of independent random walkers (IRW). In this case the generator is

$$L_{\mathrm{IRW}} f(\eta) = \sum_{i\in\mathbb{Z}} \left(\eta_i \nabla_{i,i+1} + \eta_{i+1}\nabla_{i+1,i}\right) f(\eta), \tag{24}$$

which implies, comparing with the reference process (16), that

$$\alpha = 2, \qquad \theta = 0. \tag{25}$$

In this process we have self-duality with single-site self-duality function:

$$d_{\mathrm{IRW}}(m,n) = \frac{n!}{(n-m)!}\, \mathbf{1}_{\{m\le n\}}. \tag{26}$$

As before, this self-duality property with self-duality function (26) continues to hold when $\theta = 0$ for all other values of $\alpha > 0$.

2.5 Main results

We state now our main results. Without loss of generality, as it was done in Definition 2.5, we will always choose in the following $\alpha = 1$, where α is the parameter defined in (4). The case of general α just corresponds to a rescaling of time, i.e. $t' = \alpha t$ (cf. Remark 2.6).

2.5.1 Exact solution of the two-particle dynamics

We start by providing the formula for the Fourier-Laplace transform of the transition probability of the distance and sum coordinates.

Theorem 2.8 (Fourier-Laplace transform for the distance and sum coordinates)**.** *Under Condition 2.2, the Fourier-Laplace transform in Definition 2.5 is given by*

$$G^{(\theta)}(w,w',\kappa,\lambda) = \frac{f^{(\theta)}_{\lambda,\kappa}(w,w')}{\mathscr{Z}^{(0)}_{\lambda,\kappa}} \left\{ \zeta_{\lambda,\kappa}^{|w'-w|-1} + \zeta_{\lambda,\kappa}^{w'+w-1}\left(2\,\frac{\mathscr{Z}^{(0)}_{\lambda,\kappa}}{\mathscr{Z}^{(\theta)}_{\lambda,\kappa}} - 1 \right) \right\}, \tag{27}$$

with

$$f^{(\theta)}_{\lambda,\kappa}(w,w') = \begin{cases} \frac{\theta\nu_\kappa^{-1}\zeta_{\lambda,\kappa}+1}{2} & \textit{if} \quad w=0, w'=0 \\ \\ \frac{\theta+1}{2} & \textit{if} \quad w\geq 1, w'=0 \\ \\ 1 & \textit{if} \quad w\geq 1, w'\geq 1, \end{cases} \tag{28}$$

and

$$\mathscr{L}^{(\theta)}_{\lambda,\kappa} \quad = \quad \nu_\kappa(\zeta^{-2}_{\lambda,\kappa}-1)+2\theta(x_{\lambda,\kappa}-\nu_\kappa), \tag{29}$$

where

$$\zeta_{\lambda,\kappa} := \zeta\left(x_{\lambda,\kappa}\right) = x_{\lambda,\kappa} - \sqrt{x^2_{\lambda,\kappa}-1}, \qquad x_{\lambda,\kappa} := \frac{1}{\nu_\kappa}\left(1+\frac{\lambda}{2}\right), \tag{30}$$
$$\nu_\kappa = \cos(\kappa) - i(p-q)\,\sin(\kappa).$$

Remark 2.9 (Meaning of ν_k and $\zeta(x)$)**.** *One recognizes that ν_κ is the Fourier transform of the increments of the discrete time asymmetric random walk on $\mathbb{Z}$ moving with probability p to the right and q to the left. Furthermore, as it will be clear from the proof of Theorem 2.8, the function $\zeta(x)$ appearing in* (30) *is related to the probability generating function of S_0, the first hitting time of the origin* 0 *of the discrete time asymmetric random walk moving with probability p to the right and q to the left starting at 1 at time zero. More precisely*

$$\zeta(x) = \mathbb{E}_1(x^{-S_0}), \qquad\qquad x\geq 1\ . \tag{31}$$

In order to give more intuition for the formula in Theorem 2.8, we transform to leftmost and rightmost position coordinates. In this coordinates the comparison between the interacting ($\theta\neq 0$) and non-interacting ($\theta=0$) case becomes more transparent. Let $(x(t),y(t))$ be the coordinates defined by

$$x(t) := \min\{x_1(t),x_2(t)\} \qquad y(t) := \max\{x_1(t),x_2(t)\}, \tag{32}$$

where $(x_1(t),x_2(t))$ denote the particle positions. We define the Laplace transform

$$\Pi^{(\theta)}((x,y),(x',y');\lambda) := \int_0^\infty \pi_t\left((x,y),(x',y')\right)e^{-\lambda t}dt, \tag{33}$$

where π_t denotes the transition probability of the process $\{(x(t),y(t)) : t\geq 0\}$ started from $(x,y)\in\mathbb{Z}\times\mathbb{Z}$.

Corollary 2.10 (Positions of leftmost and rightmost particle)**.** *Under Condition 2.2 and assuming $x\neq y$, the Laplace transform in* (33) *is given by*

$$\Pi^{(\theta)}((x,y),(x',y');\lambda) = \begin{cases} A^{(\theta)}_+(x'-x,y'-y,\lambda)+A^{(\theta)}_-(y'-x,x'-y,\lambda) & \textit{if} \quad y'>x' \\ A^{(\theta)}_0(x'-x,x'-y,\lambda) & \textit{if} \quad y'=x' \end{cases} \tag{34}$$

where

$$A^{(\theta)}_{\pm,0}(x,y,\lambda) := \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\Gamma^{(\theta)}_{\pm,0}(\frac{\kappa_1+\kappa_2}{2},\lambda)\, e^{i(\kappa_1 x+\kappa_2 y)}}{1+\frac{\lambda}{2}-\left(\cos(\frac{\kappa_2+\kappa_1}{2})-i(p-q)\sin\left(\frac{\kappa_2+\kappa_1}{2}\right)\right)\cos(\frac{\kappa_2-\kappa_1}{2})}\, d\kappa_1\, d\kappa_2 \tag{35}$$

and

$$\Gamma^{(\theta)}_{+}(\kappa,\lambda) := 1, \qquad \Gamma^{(\theta)}_{-}(\kappa,\lambda) := 2\,\frac{\mathscr{Z}^{(0)}_{\lambda,\kappa}}{\mathscr{Z}^{(\theta)}_{\lambda,\kappa}} - 1, \qquad \Gamma^{(\theta)}_{0}(\kappa,\lambda) := (\theta+1)\,\frac{\mathscr{Z}^{(0)}_{\lambda,\kappa}}{\mathscr{Z}^{(\theta)}_{\lambda,\kappa}}. \tag{36}$$

From the above formula we immediately see that the Fourier transform of $A^{(\theta)}_{\pm,0}(x,y,\lambda)$ is given by

$$\widehat{A}^{(\theta)}_{\pm,0}(\kappa_1,\kappa_2,\lambda) = \frac{\Gamma^{(\theta)}_{\pm,0}(\frac{\kappa_1+\kappa_2}{2},\lambda)}{2+\lambda-(\cos\kappa_1+\cos\kappa_2)+(p-q)i(\sin\kappa_1+\sin\kappa_2)}. \tag{37}$$

Notice that for $\theta = 0$ we have $\Gamma^{(0)}_{\pm,0}(\kappa,\lambda) = 1$, and thus we recover the Fourier-Laplace transform of the transition probability of two independent random walkers.

2.5.2 Scaling limits in the symmetric case

Our second main result is related to the characterization of the scaling limit of the two-particle process in the symmetric case (corresponding to the parameter p in (5) equal to 1/2). We thus consider a diffusive scaling of space and time. Being $\alpha = 1$ (cf. the beginning of Section 2.5), this leaves only $\theta > 0$ as a free parameter. Given a scaling parameter $\epsilon > 0$, we define

$$U_\epsilon(t) := \frac{\epsilon\, u(\epsilon^{-2}t)}{\sqrt{2}}, \qquad W_\epsilon(t) := \frac{\epsilon\, w(\epsilon^{-2}t)}{\sqrt{2}}\ . \tag{38}$$

We also assume that the initial values can depend on ϵ, i.e. $u_\epsilon = u(0)$ and $w_\epsilon = w(0)$, and we define

$$U := \lim_{\epsilon\to 0}\frac{\epsilon u_\epsilon}{\sqrt{2}}, \qquad W := \lim_{\epsilon\to 0}\frac{\epsilon w_\epsilon}{\sqrt{2}}. \tag{39}$$

with $U \in \mathbb{R}$ and $W \in \mathbb{R}_+$. Similarly we suppose θ to be a function of ϵ, and thus write θ_ϵ and we distinguish three different regimes as $\epsilon \to 0$:

(a) **Reflected Regime:** $\lim_{\epsilon\to 0}\epsilon\theta_\epsilon = 0$

(b) **Sticky Regime:** $\theta_\epsilon > 0$ and $\epsilon\theta_\epsilon = O(1)$. In this regime we define

$$\gamma := \lim_{\epsilon\to 0}\frac{\epsilon\theta_\epsilon}{\sqrt{2}} \in (0,\infty) \tag{40}$$

(c) **Absorbed Regime:** $\theta_\epsilon > 0$ and $\lim_{\epsilon\to 0}\epsilon\theta_\epsilon = +\infty$

We have the following result.

Theorem 2.11 (Scaling limits). *Let $\{B(t) : t \geq 0\}$ and $\{\tilde{B}(t) : t \geq 0\}$ be two independent Brownian motions starting, respectively, at $W \geq 0$ and at the origin 0. Let $s(t)$ be defined by*

$$s^{-1}(t) = t + \gamma L(t), \tag{41}$$

where $L(t)$ is the local time at the origin of $B(t)$, so that $\{B^S(t) = |B(s(t))| : t \geq 0\}$ is the one-sided sticky Brownian motion started at W, with stickiness at the origin of parameter $\gamma \in [0, \infty]$. Let $\{B^R(t) : t \geq 0\}$ denote the Brownian motion reflected at the origin started at $W \geq 0$ and let $\{B^A(t) : t \geq 0\}$ denote the Brownian motion absorbed at the origin started at $W \geq 0$. Then the following holds true: under Condition 2.2 we have

$$\lim_{\epsilon \to 0} ((U_\epsilon(t) - U), W_\epsilon(t)) = (U(t), W(t)) \tag{42}$$

where $\{(U(t), W(t)) : t \geq 0\}$ is defined by $(U(0), W(0)) = (0, W)$ and

$$(U(t), W(t)) = \begin{cases} (\tilde{B}(t), B^R(t)) & \text{in the Reflected Regime} \\ \\ (\tilde{B}(2t - s(t)), B^S(t)) & \text{in the Sticky Regime} \\ \\ (\tilde{B}(2t - t \wedge \tau_W), B^A(t)) & \text{in the Absorbed Regime} \end{cases} \tag{43}$$

where the convergence in (42) is in the sense of finite-dimensional distributions and τ_W in the third line of (43) is the absorption time of $\{B^A(t) : t \geq 0\}$.

Thus the scaling limit of the two particle process turns out to be two Brownian motions with "sticky interaction", that can be thought of as an interpolation between two coalescing Brownian motions and two reflecting Brownian motions. More precisely, the distance between the particles converges to a sticky Brownian motion, which in turn has two limiting cases, namely the absorbed and reflected Brownian motion. On the other hand, the sum of the particle positions becomes a process which is subjected to the sticky Brownian motion driving the difference and is "moving at faster rate" when the particles are together, i.e., it is a time-changes Brownian motion of which the clock runs faster with an acceleration determined by the local intersection time.

Remark 2.12 (The symmetric inclusion process in the condensation regime). *For the symmetric inclusion process SIP(k) we say that we are in the condensation regime when the parameter k tends to zero sufficiently fast, i.e., when the spreading of the particles is much slower than the attractive interaction due to the inclusion jumps [11, 1]. After a suitable rescaling two SIP(k) particles will then behave as independent Brownian motions which spend "excessive" local time together. The* sticky regime *with stickiness parameter $\gamma \in (0, \infty)$ corresponds to the choice $k = \frac{\epsilon}{\gamma\sqrt{2}}$, and acceleration of time by a factor $\epsilon^{-3}\frac{\gamma}{\sqrt{2}}$. I.e., this corresponds to the condensation regime $k \to 0$, where time is diffusively rescaled, and speeded up with an extra ϵ^{-1} in order to compensate for the vanishing diffusion rate.*

Remark 2.13 (Exclusion particles scale to reflected Brownian motions). *For the exclusion process SEP(j) it is not possible to consider the sticky or the absorbed regime, because $\theta < 0$. For this reason we only scale time diffusively with a factor $2j\epsilon^{-2}$ and take $\theta = -1/j$ fixed. This then corresponds to consider the reflected regime in (42) where $(U_\epsilon(t) - U, W_\epsilon(t))$ converge to $(\tilde{B}(t), |B(t)|)$ where $\tilde{B}(t)$ is a standard Brownian motion and $B(t)$ is an independent Brownian motion started at W.*

Next we show that the expected local time of the difference process $\{W_\epsilon(t), t \geq 0\}$ converges to the expected local time of the limiting sticky Brownian motion (in the sense of convergence of the Laplace transform). Notice that this does not follow from weak convergence of the previous Theorem 2.11, but has to be viewed rather as a result in the spirit of a local limit theorem.

PROPOSITION 2.14 (Local time in 0). *We have*

$$\int_0^\infty e^{-\lambda t}\, \mathbb{P}_w\left(w(t) = 0\right) dt = \zeta_\lambda^w \, \frac{1 + \theta\zeta_\lambda^{\mathbf{1}_{w=0}}}{\zeta_\lambda^{-1} + (\theta\lambda - 1)\zeta_\lambda} \tag{44}$$

with

$$\zeta_\lambda := \zeta_{\lambda,0} := 1 + \frac{\lambda}{2} - \sqrt{\lambda + \frac{\lambda^2}{4}} \tag{45}$$

As a consequence, in the Sticky regime we have

$$\lim_{\epsilon \to 0} \int_0^\infty e^{-\lambda t}\, \mathbb{P}_{w_\epsilon}\left(W_\epsilon(t) = 0\right) dt = \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda}\, e^{-\sqrt{2\lambda} W} \tag{46}$$

with W as in (39)*, γ as in* (40)*.*

REMARK 2.15. *Notice that the r.h.s. of* (46) *is exactly the Laplace transform of the probability $\mathbb{P}_W(B^S(t) = 0)$ of the sticky Brownian motion started at $W \geq 0$ to be at the origin at time t, see Lemma 4.5.*

2.5.3 Coarsening in the condensation regime of the inclusion process

We now present some results for the symmetric inclusion process SIP(k), which is a self-dual process. Let the time-dependent covariances of the particle numbers at sites $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ at time $t \geq 0$ be defined as

$$\Xi^{(\theta)}(t, x, y; \nu) = \int \mathbb{E}_\eta\left[(\eta_x(t) - \rho_x(t))(\eta_y(t) - \rho_y(t))\right] d\nu(\eta), \tag{47}$$

where ν denotes the initial measure (i.e. the initial distribution of the particle numbers) and

$$\rho_x(t) = \int \mathbb{E}_\eta\left[\eta_x(t)\right] d\nu(\eta). \tag{48}$$

The following Theorem gives an explicit result for the variance and the covariance of the time-dependent particle numbers in the *sticky regime* of the symmetric inclusion process when starting from a homogeneous product measure in infinite volume. In particular, we see how the variance diverges when the inclusion parameter $k = 1/\theta$ goes to zero, which corresponds to the condensation limit with piling up of particles.

THEOREM 2.16 (Scaling of variance and covariances in the sticky regime of the inclusion process). *Let $\{\eta(t) : t \geq 0\}$ be the reference process with generator* (16) *with $\alpha = 1$ and $\theta > 0$ (i.e. the time rescaled inclusion process, see* (19)*). Suppose we are in the Sticky Regime, i.e. $\epsilon\theta_\epsilon = O(1)$ as $\epsilon \to 0$. Let γ as in* (40) *and let $a > 0$. Then, for any initial homogeneous product measure ν, we have:*

a) Scaling of covariances. *If $x \neq y$, then*

$$\int_0^\infty e^{-\lambda t}\, \Xi^{(\theta_\epsilon)}(\epsilon^{-a}t, \lfloor x\epsilon^{-1}\rfloor, \lfloor y\epsilon^{-1}\rfloor; \nu)\, dt = \begin{cases} -\frac{\epsilon^{\frac{a}{2}-1}}{\sqrt{2}\gamma\lambda}\, e^{-\sqrt{\lambda}|x-y|\epsilon^{\frac{a}{2}-1}}(1+o(1)) & \text{for } 1<a<2, \\ -\frac{\gamma\rho^2\, e^{-\sqrt{\lambda}|x-y|}}{\sqrt{2\lambda}+\gamma\lambda}\,(1+o(1)) & \text{for } a=2, \\ -\frac{\gamma\rho^2}{\sqrt{2\lambda}}\, \epsilon^{\frac{a}{2}-1}(1+o(1)) & \text{for } a>2. \end{cases} \tag{49}$$

b) Scaling of variance. *If $x = y$, then*

$$\int_0^\infty e^{-\lambda t}\, \Xi^{(\theta_\epsilon)}(\epsilon^{-a}t, \lfloor x\epsilon^{-1}\rfloor, \lfloor x\epsilon^{-1}\rfloor; \nu)\, dt = \begin{cases} \frac{2\rho^2}{\lambda\sqrt{\lambda}}\, \epsilon^{-\frac{a}{2}}(1+o(1)) & \text{for } 1<a<2, \\ \frac{2\sqrt{2}\gamma\rho^2}{2\lambda+\gamma\lambda\sqrt{2\lambda}}\, \epsilon^{-1}\,(1+o(1)) & \text{for } a=2, \\ \frac{\sqrt{2}\,\gamma\rho^2}{\lambda}\, \epsilon^{-1}(1+o(1)) & \text{for } a>2. \end{cases} \tag{50}$$

Remark 2.17 (Coarseing)**.** *We see that the r.h.s. of* (49) *has three different regimes which can intuitively be understood as follows.*

1. Subcritical time scale. *In the first regime, corresponding to "short times" we see that the covariance goes to zero as $\epsilon \to 0$, which is a consequence of the initial product measure structure. At the same time we see a scaling corresponding to the Laplace transform of expected local intersection time of coalescing Brownian motions (cf. limit of $\gamma \to \infty$ of* (46)*.) This corresponds to the dynamics of large piles (at typical distance ϵ^{-1}) which merge as coalescing Brownian motions, because on the time scale under consideration there is no possibility to detach.*

2. Critical time scale. *In the second regime corresponding to "intermediate times" we see a scaling corresponding to the Laplace transform of expected local intersection time of sticky Brownian motions. This identifies the correct scale at which the "piles" have a non-trivial dynamics, i.e., can interact, merge and detach. This is also the correct time scale for the density fluctuation field (cf. Theorem 2.18).*

3. Supercritical time scale. *In the last regime, the covariance is $o(1)$ as $\epsilon \to 0$, which corresponds to the stationary regime, in which again a product measure is appearing. The $1/\sqrt{\lambda}$ scaling corresponds in time variable to $1/\sqrt{t}$, which corresponds to the probability density of two independent Brownian motions, initially at $\epsilon^{-1}x, \epsilon^{-1}y$ to meet after a time $\epsilon^{-a}t$, indeed:*

$$\frac{\exp\left\{-\frac{(x-y)^2\epsilon^{-2}}{2t\epsilon^{-a}}\right\}}{\sqrt{2\pi t\epsilon^{-a}}} \approx \frac{\epsilon^{a/2}}{\sqrt{2\pi t}}$$

This corresponds to the fact that on that longer time scale, the stickyness of the piles disappears and they move as independent particles, unless they are together.

2.5.4 Variance of the density field in the condensation regime of the inclusion process

Having identified the relevant scaling of the variance and covariance of the time-dependent particle number, we apply this to compute the limiting variance of the rescaled density fluctuation field, which shows a non-trivial limiting dependence structure in space and time. We consider the density fluctuation field out of equilibrium, i.e., we start the process from an homogeneous invariant product measure ν which is not the stationary distribution and has expected particle number $\int \eta_x d\nu = \rho$ for all $x \in \mathbb{Z}$. More precisely, we study the behavior of the random time-dependent distribution which is defined by its action on a Schwarz function $\Phi : \mathbb{R} \to \mathbb{R}$ via

$$\mathscr{X}_\epsilon(\Phi, \eta, t) = \epsilon \sum_{x\in\mathbb{Z}} \Phi(\epsilon x)(\eta_x(\epsilon^{-2}t) - \rho) \tag{51}$$

where η in the l.h.s. of (51) refers to the intitial configuration $\eta(0)$ which is distributed according to ν. Notice that we multiply by ϵ in (51) as opposed to the more common $\sqrt{\epsilon}$ which typically appears in fluctuation fields of particle systems (in dimension one) with a conserved quantity and which then usually converges to an infinite dimensional Ornstein-Uehlenbeck process see e.g. [13], Chapter 11. Here, on the contrary, we are in the condensation regime, and therefore the variance of the particle occupation numbers is of order ϵ^{-1} by Theorem 2.16, which explains why we have to multiply with an additional factor $\sqrt{\epsilon}$ in comparison with the standard setting.

Theorem 2.18 (Variance of the density fluctuation field)**.** *Let $\{\eta(t) : t \geq 0\}$ be the reference process with generator* (16) *with $\alpha = 1$ and $\theta > 0$ (i.e. the time rescaled inclusion process, see* (19)*). Assume we are in the sticky regime, i.e. $\theta = \epsilon\theta_\epsilon = \mathscr{O}(1)$ as $\epsilon \to 0$ and let γ as in* (40)*. Let ν be an initial homogeneous product measure then*

$$\lim_{\epsilon\to 0} \int_0^\infty e^{-\lambda t}\, \mathbb{E}_\nu\big[\,(\mathscr{X}_\epsilon(\Phi,\eta,t))^2\,\big]\, dt =$$
$$= \frac{\gamma\rho^2}{\sqrt{2\lambda}+\gamma\lambda}\int \Phi(x)\Phi(y)\, e^{-\sqrt{\lambda}|x-y|}\, dx\, dy + \frac{2\sqrt{2}\,\gamma\rho^2}{2\lambda+\gamma\lambda\sqrt{2\lambda}}\int \Phi(x)^2\, dx. \tag{52}$$

The limiting variance of the density fluctuation field consists of two terms which both contain the stickyness parameter γ. The combination of both terms describe how from the initial homogeneous measure ν one enters the condensation regime. Comparing to the standard case of e.g. independent random walkers, we have to replace $\gamma\rho^2$ by ρ in the numerator and replace γ by zero in the denominator. Then we exactly recover the variance of the non-stationary density fluctuation field of a system of independent walkers starting from ν. So we see that the stickyness introduces a different time dependence of the variance visible in the extra λ-dependent terms in the denominators of the r.h.s. of (52). In particular, in the first term on the r.h.s. of (52) we recognize the Laplace transform of the expected local time of sticky Brownian motion.

2.6 Discussion

In this section, we discuss relations to the literature, possible extensions and open problems.

Other applications of Theorem 2.8. In a forthcoming work [7] by the first and third author of this paper, in collaboration with M. Jara, the formula for the Laplace-Fourier transform of the transition probability of distance and sum coordinates is applied to obtain the second order Boltzmann Gibbs principle, which is a crucial ingredient in the proof of Kardar-Parisi-Zhang behavior for the weakly asymmetric inclusion process.

Dependence structure and type of convergence. The difference and sum processes, with generators (8) and (9), have a dependence structure similar to their scaling limits in Theorem 2.11. Namely, one process is autonomous, the other is depending on the first via a local time. In the scaling limit one further introduces an additional time-change that however does not change such dependence structure. The scaling limit result in Theorem 2.11 is proved in the sense of finite-dimensional distributions. One could get a stronger type of convergence by directly studying the scaling limits of the generators of the distance and sum process. This however poses additional difficulties and is not pursued here.

Scaling limits to sticky Brownian motions. We observed in Remark 2.13 that exclusion particles always scale to reflected Brownian motions. In [18] Rácz and Shkolnikov obtain multidimensional sticky Brownian motions as limits of exclusion processes. However this result is proved for a *modified* exclusion process, in which particles slow down their velocities whenever two or more particles occupy adjacent sites. Under diffusive scaling of space and time this slowing down results into a stickiness and the process converge to sticky Brownian motion in the wedge [18].

Dualities. In this paper we have focused on self-duality of particle systems. However, the same strategy would apply to interacting diffusions that are *dual* to particle systems. For instance, there are processes such as the Brownian Momentum Process [10], the Brownian Energy Process [9] and the Asymmetric Brownian Energy Process [6], which are dual to the symmetric inclusion process SIP(k). As a consequence all the results derived in this paper for the symmetric inclusion process can also be directly translated into results for these processes.

Fluctuation field in the condensation regime. As far as we know, our result is the first computation dealing with the fluctuation field of the symmetric inclusion process in the condensation regime. We conjecture the expression we have found for the variance of the fluctuation field in Theorem 2.18 to have some degree of universality within the realm of system exhibiting condensation effects. Namely, we believe that the scaling behaviour of the density field in the condensation regime, and in particular the appearance of sticky Brownian motion, is generic for systems with condensation and goes beyond systems with self-duality, e.g. including zero range processes with condensation.

Asymmetric processes. The main formula in Theorem 2.8 include interacting particle systems with asymmetries of certain type, for instance naive asymmetry (see Condition 2.2).

One could then repeat the analysis of the scaling limit of the two-particle process. In the weak-asymmetry limit one then expects sticky Brownian motions with drift as limiting processes. The possibility to apply the exact formula to asymmetric systems with duality is instead unclear, since in the presence of naive asymmetry self-duality is lost. One may hope to derive a more general formula for the two particle dynamics that would apply to systems with asymmetry and self-duality such as ASEP(q,j) [5], ASIP(q,k) [6], ABEP(k) [6] processes.

2.7 Organization of the paper.

The rest of this paper is organized as follows. Section 3 contains the proof of Theorem 2.8 on the Laplace-Fourier transform of the transition probability of the distance and sum coordinates. In Section 4 we prove Theorem 2.11 on the scaling limits of the two particle process. In Section 5 we prove applications for particle systems with self-duality. We first prove the scaling behavior of the variance and covariances of the particle occupation number for the inclusion process in the condensation regime (Theorem 2.16). Then we prove the scaling behavior for the variance of the density field in the same regime (Theorem 2.18).

3 Two-particle dynamics: proof of Theorem 2.8

3.1 Outline of the proof

The strategy to solve the two particle dynamics has two steps: first we analyze the autonomous distance process, for which the main challenge is to treat the spatial inhomogeneity caused by the defect in 1; second we study the sum process by conditioning to the distance.

Since $u(t)$ and $w(t)$ jump at the same times, we can define the process $N(t)$ that gives the number of jumps of $(u(t), w(t))$ up to time $t \geq 0$. Notice that for any $t \geq 0$, $u(t) + w(t) \in w + 2\mathbb{Z}$. Given the trajectory $\{w(t),\, t \geq 0\}$, $N(t)$ is a Poisson process with time-dependent intensity which has the following (time-dependent) generator:

$$Lf(n) = 2\left(1 + \frac{\theta}{2}\,\mathbf{1}_{w_t=1}\right)[f(n+1) - f(n)]\,. \tag{53}$$

In the following proposition we obtain a formula for $G(w, w'\kappa, \lambda)$ in terms of the Poisson clock $N(t)$ exploiting the fact that, conditioned to the path $\{w(t), t \geq 0\}$ the process $u(t)$ performs a standard discrete-time asymmetric random walk, for which we know the characteristic function at any time.

Proposition 3.1. *We have*

$$G(w, w', \kappa, \lambda) = \int_0^\infty g_\kappa(w, w', t)\, e^{-\lambda t}\, dt, \tag{54}$$

with

$$g_\kappa(w, w', t) := \mathbb{E}_w\left[\mathbf{1}_{w(t)=w'}\; \nu_\kappa^{N(t)}\right], \qquad \nu_\kappa := \cos(\kappa) - i\,(p-q)\,\sin(\kappa). \tag{55}$$

Proof. We have that

$$
\begin{aligned}
G(w,w',\kappa,\lambda) &= \sum_{u'\in\mathbb{Z}} e^{-i\kappa u'}\mathscr{G}\left((0,w);(u',w');\lambda\right) \\
&= \int_0^\infty e^{-\lambda t}\left(\sum_{u'\in\mathbb{Z}} P_t\left((0,w);(u',w')\right)\, e^{-i\kappa u'}\right) dt.
\end{aligned}
$$

Then we need to prove that

$$
\sum_{u'\in\mathbb{Z}} P_t\left((0,w);(u',w')\right)\, e^{-i\kappa u'} = g_\kappa(w,w',t), \tag{56}
$$

with $g_\kappa(w,w',t)$ as in (55). For $\kappa\in\mathbb{R}$ we have

$$
\begin{aligned}
&\sum_{u'\in\mathbb{Z}} P_t\left((0,w);(u',w')\right)\, e^{-i\kappa u'} = \\
&= \sum_{u'\in\mathbb{Z}} \mathbb{E}_w\Big[P_t\Big((0,w);(u',w')\ \Big|\ \{w(s), 0\le s\le t\}\Big)\Big]\, e^{-i\kappa u'} \\
&= \mathbb{E}_w\Big[\mathbf{1}_{w(t)=w'}\cdot\sum_{u'\in\mathbb{Z}}\mathbb{P}\Big(u(t)=u'\Big|u(0)=0,\{w(s),0\le s\le t\}\Big)\cdot e^{-i\kappa u'}\Big].
\end{aligned}
\tag{57}
$$

Let us denote by $p^{(n)}(u,u')$ the n-steps transition probability function of the asymmetric discrete-time random walk that jumps to the right with probability p and to the left with probability $q=1-p$. Then we have

$$
\sum_{u\in\mathbb{Z}} p^{(1)}(0,u)\, e^{-i\kappa u} = qe^{i\kappa}+pe^{-i\kappa} = \cos(\kappa)-i(p-q)\sin(\kappa)=\nu_\kappa,
$$

and

$$
\sum_{u\in\mathbb{Z}} p^{(n)}(0,u)\, e^{-i\kappa u} = \nu_\kappa^n. \tag{58}
$$

According to (9), the conditioned process $\{u(t)\,|\,\{w(t),t\ge 0\}\}$ is equivalent to $\{u(t)\,|\,N(t)\}$ that reduces to a discrete time random walk on $\mathbb{Z}$ that jumps to the right with probability p and to the left with probability $q=1-p$. Thus

$$
\begin{aligned}
&\sum_{u'\in\mathbb{Z}}\mathbb{P}\Big(u(t)=u'\Big|u(0)=0,\{w(s),0\le s\le t\}\Big)\cdot e^{-i\kappa u'} \\
&= \sum_{u'\in\mathbb{Z}} e^{-i\kappa u'} p^{(N(t))}(0,u') \\
&= \nu_\kappa^{N(t)}.
\end{aligned}
$$

Then (56) follows from (57). $\square$

In the following we obtain a convolution equation for $g_\kappa(w,w',t)$ by conditioning on the first hitting time of the defective site 1. We distinguish several cases, depending on whether w and w' are equal to 0, 1, or larger than 1. When the process is at the right of 1 it can be treated as a standard random walk. This produces a system of linear equations for $G_\kappa(\cdot,\cdot,\lambda)$ that can easily be solved.

3.2 Case $w' = 0$

1. **Case $\mathbf{w \geq 2}$**. Denote by T_1 the first hitting time of 1 and by $f_{T_1,w}$ its probability density when the walk starts from w. From (53) it is clear that, for $w \geq 2$, $N(t)$ behaves as a Poisson process with rate 2 up to time T_1, then

$$\mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\big|T_1\right] = e^{2(\nu_\kappa - 1)T_1}. \tag{59}$$

Hence, denoting by $\mathscr{F}_{T_1}$ the pre-T_1 sigma-algebra of the process $w(t)$,

$$\begin{aligned}
g_\kappa(w,0,t) \quad &= \mathbb{E}_w\left[\mathbf{1}_{w(t)=0} \;\cdot \nu_\kappa^{N(t)}\right] \\
&= \mathbb{E}_w\left[\mathbb{E}_w\left[\mathbf{1}_{w(t)=0}\; \nu_\kappa^{N(t)}\Big|\mathscr{F}_{T_1}\right]\right] \\
&= \mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\; \mathbb{E}_w\left[\mathbf{1}_{w(t)=0}\; \nu_\kappa^{N(t)-N(T_1)}\Big|\mathscr{F}_{T_1}\right]\right] \\
&= \mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\; \mathbb{E}_1\left[\mathbf{1}_{w(t-T_1)=0}\; \nu_\kappa^{N(t-T_1)}\right]\right] \\
&= \mathbb{E}_w\left[\mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\; g_\kappa(1,0,t-T_1)\Big|T_1\right]\right] \\
&= \mathbb{E}_w\left[g_\kappa(1,0,t-T_1)\,\mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\big|T_1\right]\right] \\
&= \int_0^t g_\kappa(1,0,t-s)\, f_{T_1,w}(s)\,\mathbb{E}_w\left[\nu_\kappa^{N(s)}\big|T_1 = s\right] ds.
\end{aligned}$$

As a consequence

$$g_\kappa(w,0,t) = \left[(h_0 \cdot f_{T_1,w}) * g_\kappa(1,0,\cdot)\right](t), \qquad h_0(t) = \mathbb{E}_w\left[\nu_\kappa^{N(t)}\big|T_1 = t\right]. \tag{60}$$

From the convolution equation (60) it follows that

$$G_\kappa(w,0,\lambda) = \Psi_w(\lambda)\cdot G_\kappa(1,0,\lambda), \qquad \text{for any } \; w \geq 2, \tag{61}$$

where

$$\begin{aligned}
\Psi_w(\lambda) \;\; &:= \;\; \int_0^\infty \mathbb{E}_w\left[\nu_\kappa^{N(t)}\big|T_1 = t\right] f_{T_1,w}(t)\, e^{-\lambda t}\, dt \\
&= \;\; \mathbf{E}_w^{\mathrm{IRW}(2)}\left[e^{-\lambda T_1}\; \nu_\kappa^{N(T_1)}\right]
\end{aligned} \tag{62}$$

where $\mathbf{E}_w^{\mathrm{IRW}(2)}$ is the expectation w.r. to the probability law of a symmetric random walk in $\mathbb{Z}$ with hopping rate 2, starting at time 0 from $w \geq 2$.

2. **Case $\mathbf{w}=\mathbf{1}$**. Let $T_i^{\rm ex}$ be the first exit time from i. Then $T_1^{\rm ex}\sim{\rm Exp}(\theta+2)$, hence

$$
\begin{aligned}
g_\kappa(1,0,t) &= \mathbb{E}_1\left[\mathbb{E}_1\left[\mathbf{1}_{w(t)=0}\ \cdot \nu_\kappa^{N(t)}\Big|\,\mathscr{F}_{T_1^{\rm ex}}\right]\right]\\
&= \nu_\kappa\ \mathbb{E}_1\left[\mathbb{E}_{w(T_1^{\rm ex})}\left[\mathbf{1}_{w(t-T_1^{\rm ex})=0}\ \cdot \nu_\kappa^{N(t-T_1^{\rm ex})}\right]\right]\\
&= \nu_\kappa\ \mathbb{E}_1\left[g_\kappa(w(T_1^{\rm ex}),0,t-T_1^{\rm ex})\right]\\
&= \nu_\kappa\ \left\{\frac{\theta+1}{\theta+2}\,\mathbb{E}_1\left[g_\kappa(0,0,t-T_1^{\rm ex})\right]+\frac{1}{\theta+2}\,\mathbb{E}_1\left[g_\kappa(2,0,t-T_1^{\rm ex})\right]\right\}\\
&= \nu_\kappa\ \int_0^t\left\{\frac{\theta+1}{\theta+2}\,g_\kappa(0,0,t-s)+\frac{1}{\theta+2}\,g_\kappa(2,0,t-s)\right\}\,(\theta+2)\,e^{-(1+\gamma)s}\,ds.
\end{aligned}
$$

Thus

$$
g_\kappa(1,0,t)=(\theta+1)\nu_\kappa[h_1*g_\kappa(0,0,\cdot)](t)+\nu_\kappa[h_1*g_\kappa(2,0,\cdot)](t),\qquad \text{with}\quad h_1(t):=e^{-(\theta+2)t}.
$$

Then we find

$$
G_\kappa(1,0,\lambda)=\frac{\nu_\kappa}{\theta+2+\lambda}\,\left[(\theta+1)G_\kappa(0,0,\lambda)+G_\kappa(2,0,\lambda)\right]. \tag{63}
$$

3. **Case $\mathbf{w}=\mathbf{0}$**. Now we have $T_0^{\rm ex}\sim{\rm Exp}(2)$ then

$$
\begin{aligned}
g_\kappa(0,0,t) &= \mathbb{E}_0\left[\mathbb{E}_0\left[\mathbf{1}_{w(t)=0}\ \cdot \nu_\kappa^{N(t)}\Big|\,\mathscr{F}_{T_0^{\rm ex}}\right]\right]\\
&= \mathbb{E}_0\left[\mathbf{1}_{T_0^{\rm ex}>t}\,\mathbb{P}_0\left[w(t)=0\Big|\,\mathscr{F}_{T_0^{\rm ex}}\right]\right]+\nu_\kappa\,\mathbb{E}_0\left[\mathbf{1}_{T_0^{\rm ex}\le t}\,\mathbb{E}_1\left[\mathbf{1}_{w(t-T_0^{\rm ex})=0}\ \cdot \nu_\kappa^{N(t-T_0^{\rm ex})}\right]\right]\\
&= \mathbb{P}_0\left(T_0^{\rm ex}>t\right)+2\nu_\kappa\int_0^t e^{-2s}\,g_\kappa(1,0,t-s)\,ds,
\end{aligned}
$$

which gives

$$
g_\kappa(0,0,t)=h_2(t)+2\nu_\kappa[h_2*g_\kappa(1,0,\cdot)](t)\qquad\text{with}\quad h_2(t)=e^{-2t}. \tag{64}
$$

Thus

$$
G_\kappa(0,0,\lambda)=\frac{1}{2+\lambda}\left(1+2\nu_\kappa G_\kappa(1,0,\lambda)\right). \tag{65}
$$

Summarizing, using (61), (63) and (65), we get

$$
\begin{aligned}
G_\kappa(0,0,\lambda) &= \frac{2+\theta+\lambda-\nu_\kappa\Psi_2(\lambda)}{\nu_\kappa\mathscr{Z}_{\lambda,\kappa}}\\
G_\kappa(1,0,\lambda) &= \frac{\theta+1}{\mathscr{Z}_{\lambda,\kappa}}\\
G_\kappa(w,0,\lambda) &= \frac{\theta+1}{\mathscr{Z}_{\lambda,\kappa}}\,\Psi_w(\lambda)\qquad\text{for}\ \ w\ge 2
\end{aligned} \tag{66}
$$

$$
\mathscr{Z}_{\lambda,\kappa}=\frac{1}{\nu_\kappa}\left\{(2+\lambda)(2+\theta+\lambda-\nu_\kappa\Psi_2(\lambda))-2(\theta+1)\nu_\kappa^2\right\} \tag{67}
$$

and $\Psi_w(\lambda)$ as in (62).

3.3 Case $w' = 1$

1. **Case $\mathbf{w \geq 2}$**. Denote by T_1 the first hitting time of 1 and, as before, by $f_{T_1,w}$ its probability density when the walk is starting from w. Then

$$
\begin{aligned}
g_\kappa(w,1,t) \qquad &= \mathbb{E}_w\left[\mathbb{E}_w\left[\mathbf{1}_{w(t)=1}\ \nu_\kappa^{N(t)}\Big|\mathscr{F}_{T_1}\right]\right] \\
&= \mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\ \mathbb{E}_1\left[\mathbf{1}_{w(t-T_1)=1}\ \nu_\kappa^{N(t-T_1)}\right]\right] \\
&= \mathbb{E}_w\left[g_\kappa(1,1,t-T_1)\,\mathbb{E}_w\left[\nu_\kappa^{N(T_1)}\big|T_1\right]\right] \\
&= \int_0^t g_\kappa(1,1,t-s)\, f_{T_1,w}(s)\,\mathbb{E}_w\left[\nu_\kappa^{N(s)}\big|T_1=s\right]ds,
\end{aligned}
$$

so that

$$g_\kappa(w,1,t) = \left[(h_0\cdot f_{T_1,w}) * g_\kappa(1,1,\cdot)\right](t). \tag{68}$$

It follows that

$$G_\kappa(w,1,\lambda) = \Psi_w(\lambda)\cdot G_\kappa(1,1,\lambda), \qquad \text{for any } \ w\geq 2. \tag{69}$$

2. **Case $\mathbf{w = 1}$**. We have

$$
\begin{aligned}
g_\kappa(1,1,t) &= \mathbb{E}_1\left[\mathbf{1}_{T_1^{\mathrm{ex}}>t}\right] + \mathbb{E}_1\left[\mathbf{1}_{T_1^{\mathrm{ex}}\leq t}\cdot\ \mathbf{1}_{w(t)=1}\ \cdot\nu_\kappa^{N(t)}\right] \\
&= \mathbb{P}_1\Big(T_1^{\mathrm{ex}}>t\Big) + \mathbb{E}_1\left[\mathbf{1}_{T_1^{\mathrm{ex}}\leq t}\cdot\ \mathbb{E}_1\left[\mathbf{1}_{w(t)=0}\ \cdot\nu_\kappa^{N(t)}\Big|\ \mathscr{F}_{T_1^{\mathrm{ex}}}\right]\right] \\
&= e^{-(2+\theta)t} + \nu_\kappa\,\mathbb{E}_1\left[\mathbf{1}_{T_1^{\mathrm{ex}}\leq t}\cdot\ g_\kappa(w(T_1^{\mathrm{ex}}),0,t-T_1^{\mathrm{ex}})\right] \\
&= e^{-(2+\theta)t} + \nu_\kappa\ \left\{\frac{(\theta+1)}{\theta+2}\,\mathbb{E}_1\left[\mathbf{1}_{T_1^{\mathrm{ex}}\leq t}\cdot\ g_\kappa(0,0,t-T_1^{\mathrm{ex}})\right] + \frac{1}{2+\theta}\,\mathbb{E}_1\left[\mathbf{1}_{T_1^{\mathrm{ex}}\leq t}\cdot\ g_\kappa(2,0,t-T_1^{\mathrm{ex}})\right]\right\} \\
&= e^{-(2+\theta)t} + \nu_\kappa\ \int_0^t\{(\theta+1)\,g_\kappa(0,0,t-s) +\ g_\kappa(2,0,t-s)\}\ e^{-(2+\theta)s}\,ds.
\end{aligned}
$$

Then

$$g_\kappa(1,1,t) = h_1(t) + \nu_\kappa\left[h_1 * ((\theta+1)g_\kappa(0,1,\cdot) + g_\kappa(2,1,\cdot))\right](t), \tag{70}$$

hence

$$G_\kappa(1,1,\lambda) = \frac{1}{2+\theta+\lambda}\ \left[1+(\theta+1)\nu_\kappa G_\kappa(0,1,\lambda) + \nu_\kappa G_\kappa(2,1,\lambda)\right]. \tag{71}$$

3. **Case $\mathbf{w = 0}$**. Now we have $T_0^{\mathrm{ex}}\sim \mathrm{Exp}(2)$. We write

$$
\begin{aligned}
g_\kappa(0,1,t) \quad &= \quad \mathbb{E}_0\left[\mathbb{E}_0\left[\mathbf{1}_{w(t)=1}\ \cdot\nu_\kappa^{N(t)}\Big|\ \mathscr{F}_{T_0^{\mathrm{ex}}}\right]\right] \\
&= \nu_\kappa\,\mathbb{E}_0\left[\mathbb{E}_1\left[\mathbf{1}_{w(t-T_0^{\mathrm{ex}})=1}\ \cdot\nu_\kappa^{N(t-T_0^{\mathrm{ex}})}\right]\right] \\
&= 2\nu_\kappa\int_0^t e^{-2s}\,g_\kappa(1,1,t-s)\,ds,
\end{aligned}
$$

which implies

$$g_\kappa(0,1,t) = 2\nu_\kappa [h_2 * g_\kappa(1,1,\cdot)](t). \tag{72}$$

Then

$$G_\kappa(0,1,\lambda) = \frac{2\nu_\kappa}{2+\lambda}\, G_\kappa(1,1,\lambda). \tag{73}$$

Thus, using (69), (71) and (73) we get

$$\begin{aligned} G_\kappa(0,1,\lambda) &= \frac{2}{\mathscr{Z}_{\lambda,\kappa}} \\ G_\kappa(1,1,\lambda) &= \frac{2+\lambda}{\nu_\kappa \mathscr{Z}_{\lambda,\kappa}} \\ G_\kappa(w,1,\lambda) &= \frac{2+\lambda}{\nu_\kappa \mathscr{Z}_{\lambda,\kappa}}\, \Psi_w(\lambda) \qquad \text{for } w \geq 2. \end{aligned} \tag{74}$$

3.4 Case $w' \geq 2$

1. **Case $\mathbf{w \geq 2}$**. Denoting by T_1 the first hitting time of 1, we have

$$\begin{aligned} g_\kappa(w,w',t) \quad &= \mathbb{E}_w\Big[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'} \ \cdot \nu_\kappa^{N(t)}\Big] + \mathbb{E}_w\Big[\mathbf{1}_{T_1\leq t}\, \mathbf{1}_{w(t)=w'} \ \cdot \nu_\kappa^{N(t)}\Big] \\ &= \mathbf{E}_w^{\mathrm{IRW}(2)}\Big[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'} \ \cdot \nu_\kappa^{N(t)}\Big] + \mathbb{E}_w\Big[\mathbf{1}_{T_1\leq t}\, \mathbb{E}_w\Big[\mathbf{1}_{w(t)=w'}\, \nu_\kappa^{N(t)}\Big|\mathscr{F}_{T_1}\Big]\Big] \\ &= \mathbf{E}_w^{\mathrm{IRW}(2)}\Big[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'} \ \cdot \nu_\kappa^{N(t)}\Big] + \int_0^t g_\kappa(1,w',t-s)\, f_{T_1,w}(s)\, \mathbb{E}_w\big[\nu_\kappa^{N(s)}\big|T_1=s\big]\, ds, \end{aligned}$$

which is equivalent to

$$g_\kappa(w,w',t) = \mathbf{E}_w^{\mathrm{IRW}(2)}\Big[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'} \ \cdot \nu_\kappa^{N(t)}\Big] + \big[(h_0 \cdot f_{T_1,w}) * g_\kappa(1,w',\cdot)\big]\,(t), \tag{75}$$

where $\mathbf{E}_w^{\mathrm{IRW}(2)}$ is the expectation w.r. to the probability law of a symmetric random walk in $\mathbb{Z}$ with hopping rate 2, starting at time 0 from $w \geq 2$.

From the convolution equation (75) it follows that

$$G_\kappa(w,w',\lambda) = \Phi_{w,w'}(\lambda) + \Psi_w(\lambda)\cdot G_\kappa(1,w',\lambda), \qquad \text{for any } w \geq 2, \tag{76}$$

where

$$\Phi_{w,w'}(\lambda) := \int_0^\infty \mathbf{E}_w^{\mathrm{IRW}(2)}\Big[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'} \ \cdot \nu_\kappa^{N(t)}\Big]\, e^{-\lambda t}\, dt. \tag{77}$$

2. **Case $\mathbf{w=1}$**. We have

$$\begin{aligned}
g_\kappa(1,w',t) \quad &= \mathbb{E}_1\Big[\mathbf{1}_{w(t)=w'}\ \cdot \nu_\kappa^{N(t)}\Big] \\
&= \mathbb{E}_1\Big[\mathbb{E}_1\Big[\mathbf{1}_{w(t)=w'}\ \cdot \nu_\kappa^{N(t)}\Big|\mathscr{F}_{T_1^{\mathrm{ex}}}\Big]\Big] \\
&= \nu_\kappa\, \mathbb{E}_1\Big[g_\kappa(w(T_1^{\mathrm{ex}}),w',t-T_1^{\mathrm{ex}})\Big] \\
&= \nu_\kappa\ \left\{\frac{\theta+1}{\theta+2}\,\mathbb{E}_1\Big[g_\kappa(0,w',t-T_1^{\mathrm{ex}})\Big]+\frac{1}{\theta+2}\,\mathbb{E}_1\Big[g_\kappa(2,w',t-T_1^{\mathrm{ex}})\Big]\right\} \\
&= \nu_\kappa\ \int_0^t \big\{(\theta+1)\,g_\kappa(0,w',t-s)+g_\kappa(2,w',t-s)\big\}\ e^{-(\theta+2)s}\,ds,
\end{aligned}$$

i.e.

$$g_\kappa(1,w',t)=(\theta+1)\nu_\kappa[h_1*g_\kappa(0,w',\cdot)](t)+\nu_\kappa[h_1*g_\kappa(2,w',\cdot)](t). \tag{78}$$

Then

$$G_\kappa(1,w',\lambda)=\frac{1}{2+\theta+\lambda}\ \big[(\theta+1)\nu_\kappa G_\kappa(0,w',\lambda)+\nu_\kappa G_\kappa(2,w',\lambda)\big]\,. \tag{79}$$

3. **Case $\mathbf{w=0}$**. Now we have $T_0^{\mathrm{ex}}\sim\mathrm{Exp}(2)$ then

$$\begin{aligned}
g_\kappa(0,w',t) \quad &= \mathbb{E}_0\Big[\mathbb{E}_0\Big[\mathbf{1}_{w(t)=w'}\ \cdot \nu_\kappa^{N(t)}\Big|\mathscr{F}_{T_0^{\mathrm{ex}}}\Big]\Big] \\
&= \nu_\kappa\,\mathbb{E}_0\Big[\mathbb{E}_1\Big[\mathbf{1}_{w(t-T_0^{\mathrm{ex}})=w'}\ \cdot \nu_\kappa^{N(t-T_0^{\mathrm{ex}})}\Big]\Big] \\
&= 2\nu_\kappa \int_0^t e^{-2s}\,g_\kappa(1,w',t-s)\,ds,
\end{aligned}$$

namely

$$g_\kappa(0,w',t)=2\nu_\kappa[h_2*g_\kappa(1,w',\cdot)](t). \tag{80}$$

Then

$$G_\kappa(0,w',\lambda)=\frac{2\nu_\kappa}{2+\lambda}\,G_\kappa(1,w',\lambda). \tag{81}$$

Thus, using (76), (79) and (81) we get

$$\begin{aligned}
G_\kappa(0,w',\lambda)&=\frac{2\nu_\kappa}{\mathscr{Z}_{\lambda,\kappa}}\,\Phi_{2,w'}(\lambda) \\
G_\kappa(1,w',\lambda)&=\frac{2+\lambda}{\mathscr{Z}_{\lambda,\kappa}}\,\Phi_{2,w'}(\lambda) \\
G_\kappa(w,w',\lambda)&=\Phi_{w,w'}(\lambda)+\frac{2+\lambda}{\mathscr{Z}_{\lambda,\kappa}}\,\Phi_{2,w'}(\lambda)\Psi_w(\lambda) \qquad \text{for } \ w\geq 2,
\end{aligned} \tag{82}$$

for $w'\geq 2$.

3.5 Computation of $\Psi_w(\lambda)$ and $\Phi_{w,w'}(\lambda)$

For $x \in \mathbb{C}$ we define

$$\zeta(x) := x - \sqrt{x^2 - 1} \tag{83}$$

notice that $\zeta(x) \in \mathbb{R}^+$ for any $|x| \geq 1$ and $\zeta(x) \leq 1$ for any $x \in \mathbb{R} \cap [1, +\infty)$.

Lemma 3.2. *For $w \geq 2$ we have*

$$\Psi_w(\lambda) = \zeta_{\lambda,\kappa}^{w-1} \qquad \textit{with} \qquad \zeta_{\lambda,\kappa} := \zeta\left(x_{\lambda,\kappa}\right), \qquad x_{\lambda,\kappa} := \frac{1}{\nu_\kappa}\left(1 + \frac{\lambda}{2}\right) \tag{84}$$

Proof. Let $\mathbf{E}_w^{\mathrm{IRW}(2)}$ be the expectation with respect to a symmetric random walk on $\mathbb{Z}$ with hopping rate 2, and let S_1 the first hitting time of 1 of the embedded discrete time random walk and denote by $\{X_i\}_{i\in\mathbb{N}}$ a sequence of independent exponential random variables of parameter 2. Then

$$\begin{aligned}
\Psi_w(\lambda) &= \mathbf{E}_w^{\mathrm{IRW}(2)}\left[e^{-\lambda T_1}\, \nu_\kappa^{N(T_1)}\right] = \mathbf{E}_w\left[\mathbf{E}_w\left[e^{-\lambda T_1}\, \nu_\kappa^{S_1}\middle| S_1\right]\right] \\
&= \sum_{n=1}^{\infty} \nu_\kappa^n\, \mathbf{P}_w\left(S_1 = n\right)\, \mathbf{E}_w\left[e^{-\lambda T_1}\middle| S_1 = n\right] \\
&= \sum_{n=1}^{\infty} \nu_\kappa^n\, \mathbf{P}_w\left(S_1 = n\right)\, \mathbf{E}_w\left[e^{-\lambda(X_1 + \ldots + X_n)}\right] \\
&= \sum_{n=1}^{\infty} \nu_\kappa^n\, \mathbf{P}_w\left(S_1 = n\right)\, \left(\frac{2}{2+\lambda}\right)^n \\
&= \mathbf{E}_w\left[\left(\frac{2\nu_\kappa}{2+\lambda}\right)^{S_1}\right] = \left(\zeta\left(\frac{2+\lambda}{2\nu_\kappa}\right)\right)^{w-1},
\end{aligned} \tag{85}$$

with $\zeta(x)$ as in (83). $\square$

Lemma 3.3. *For $w, w' \geq 2$ we have*

$$\Phi_{w,w'}(\lambda) = \frac{\zeta_{\lambda,\kappa}^{|w'-w|} - \zeta_{\lambda,\kappa}^{w'+w-2}}{\nu_\kappa\left(\zeta_{\lambda,\kappa}^{-1} - \zeta_{\lambda,\kappa}\right)} \tag{86}$$

with $\zeta_{\lambda,\kappa}$ as in (84).

Proof. By definition

$$\Phi_{w,w'}(\lambda) := \int_0^\infty \mathbf{E}_w^{\mathrm{IRW}(2)}\left[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'}\ \cdot \nu_\kappa^{N(t)}\right] e^{-\lambda t}\, dt \tag{87}$$

$$\mathbf{E}_w^{\mathrm{IRW}(2)}\left[\mathbf{1}_{T_1>t}\, \mathbf{1}_{w(t)=w'}\ \cdot \nu_\kappa^{N(t)}\right] = \sum_{n=0}^{\infty} \nu_\kappa^n\, \mathbf{p}_w\left(S_1 > n,\, w_n = w'\right)\, P(N(t) = n) \tag{88}$$

where now $\mathbf{p}_w$ is the probability law of the embedded symmetric random walk on $\mathbb{Z}$ starting from $w \geq 2$, and S_1 is the related first hitting time of 1. Moreover $N(t)$ is the Poisson process of parameter 2. From the reflection principle for the symmetric random walk we have that

$$\begin{aligned} &\mathbf{p}_w\left(S_1 \leq n,\, w_n = w'\right) = \mathbf{p}_0\left(S_{1-w} \leq n,\, w_n = w' - w\right) \\ &= \mathbf{p}_0\left(w_n = 2 - (w + w')\right) = \mathbf{p}_0\left(w_n = (w + w') - 2\right) \end{aligned} \tag{89}$$

where, for $b \geq 0$,

$$\mathbf{p}_0(w_n = b) = \frac{1}{2^n}\binom{n}{(n+b)/2} \quad \text{if } n \geq b \quad \text{and} \quad n + b \text{ is even} \tag{90}$$

and it is 0 otherwise. Then

$$\mathbf{E}_w^{\mathrm{IRW}(2)}\left[\mathbf{1}_{T_1>t}\,\mathbf{1}_{w(t)=w'}\;\cdot \nu_\kappa^{N(t)}\right] = e^{-2t}\sum_{n=0}^{\infty}\frac{(2\nu_\kappa t)^n}{n!}\;\left\{\mathbf{p}_0\left(w_n = w' - w\right) - \mathbf{p}_0\left(w_n = w' + w - 2\right)\right\}$$

hence, from (90) it follows

$$\begin{aligned} \Phi_{w,w'}(\lambda) &= \sum_{n=0}^{\infty}\frac{(2\nu_\kappa)^n}{n!}\;\left\{\mathbf{p}_0\left(w_n = w' - w\right) - \mathbf{p}_0\left(w_n = w' + w - 2\right)\right\}\int_0^\infty t^n\, e^{-(2+\lambda)t}\, dt \\ &= \sum_{n=0}^{\infty}\frac{(2\nu_\kappa)^n}{(2+\lambda)^{n+1}}\;\left\{\mathbf{p}_0\left(w_n = w' - w\right) - \mathbf{p}_0\left(w_n = w' + w - 2\right)\right\} \\ &= \frac{1}{2+\lambda}\;\left\{f(w' - w) - f(w' + w - 2)\right\} \end{aligned} \tag{91}$$

with, for $b \in \mathbb{Z}$,

$$f(b) \;:=\; \sum_{n=0}^{\infty}\left(\frac{2\nu_\kappa}{2+\lambda}\right)^n\;\mathbf{p}_0\left(w_n = b\right) = \frac{2+\lambda}{\sqrt{(2+\lambda)^2 - 4\nu_\kappa^2}}\;\left(\zeta\left(\frac{2+\lambda}{2\nu_\kappa}\right)\right)^{|b|}$$

and $\zeta(x)$ as in (83). $\square$

3.6 Conclusion of the proof of Theorem 2.8

From (66), (74) and (82) it follows that

$$\begin{aligned} G_\kappa(0,0,\lambda) &= \frac{\theta\nu_\kappa^{-1} + \zeta_{\lambda,\kappa}^{-1}}{\mathscr{Z}_{\lambda,\kappa}} \\ G_\kappa(w,0,\lambda) &= \frac{\theta+1}{\mathscr{Z}_{\lambda,\kappa}}\,\zeta_{\lambda,\kappa}^{w-1} \qquad \text{for } w \geq 1 \\ G_\kappa(0,w',\lambda) &= \frac{2}{\mathscr{Z}_{\lambda,\kappa}}\,\zeta_{\lambda,\kappa}^{w'-1} \qquad \text{for } w' \geq 1 \\ G_\kappa(w,w',\lambda) &= \frac{\zeta_{\lambda,\kappa}^{|w'-w|} - \zeta_{\lambda,\kappa}^{w'+w-2}}{\nu_\kappa\left(\zeta_{\lambda,\kappa}^{-1} - \zeta_{\lambda,\kappa}\right)} + \frac{2x_{\lambda,\kappa}}{\mathscr{Z}_{\lambda,\kappa}}\,\zeta_{\lambda,\kappa}^{w'+w-2} \qquad \text{for } w,w' \geq 1 \end{aligned} \tag{92}$$

with

$$\zeta_{\lambda,\kappa} := \zeta\left(x_{\lambda,\kappa}\right) = x_{\lambda,\kappa} - \sqrt{x_{\lambda,\kappa}^2 - 1}, \qquad x_{\lambda,\kappa} := \frac{1}{\nu_\kappa}\left(1 + \frac{\lambda}{2}\right) \tag{93}$$

and

$$\begin{aligned}
\mathscr{Z}_{\lambda,\kappa} &= \frac{1}{\nu_\kappa}\left\{(2+\lambda)(2+\theta+\lambda-\nu_\kappa\Psi_2(\lambda)) - 2(\theta+1)\nu_\kappa^2\right\} \\
&= \Big(\nu_\kappa(\zeta_{\lambda,\kappa}^{-2} - 1) + 2\theta\left(x_{\lambda,\kappa} - \nu_\kappa\right)\Big) \\
&= \mathscr{Z}_{\lambda,\kappa}^{(0)} + 2\theta(x_{\lambda,\kappa} - \nu_\kappa), \qquad \mathscr{Z}_{\lambda,\kappa}^{(0)} := \nu_\kappa(\zeta_{\lambda,\kappa}^{-2} - 1)
\end{aligned} \tag{94}$$

Notice that, for $w, w' \geq 1$

$$G_\kappa(w, w', \lambda) \quad = \quad \frac{1}{\mathscr{Z}_{\lambda,\kappa}^{(0)}}\left\{\zeta_{\lambda,\kappa}^{|w'-w|-1} + \zeta_{\lambda,\kappa}^{w'+w-1} \cdot \left(2\,\frac{\mathscr{Z}_{\lambda,\kappa}^{(0)}}{\mathscr{Z}_{\lambda,\kappa}} - 1\right)\right\}$$

thus

$$\begin{aligned}
&G_\kappa(0,0,\lambda) = \frac{\theta\nu_\kappa^{-1} + \zeta_{\lambda,\kappa}^{-1}}{\mathscr{Z}_{\lambda,\kappa}} \\
&G_\kappa(w,0,\lambda) = \frac{\theta+1}{\mathscr{Z}_{\lambda,\kappa}}\,\zeta_{\lambda,\kappa}^{w-1} \qquad \text{for } \ w \geq 1 \\
&G_\kappa(w,w',\lambda) = \frac{1}{\mathscr{Z}_{\lambda,\kappa}^{(0)}}\left\{\zeta_{\lambda,\kappa}^{|w'-w|-1} + \zeta_{\lambda,\kappa}^{w'+w-1} \cdot \left(2\,\frac{\mathscr{Z}_{\lambda,\kappa}^{(0)}}{\mathscr{Z}_{\lambda,\kappa}} - 1\right)\right\} \qquad \text{for } \ w \geq 0, w' \geq 1.
\end{aligned}$$

This finishes the proof of the Theorem 2.8. □

3.7 Proof of Corollary 2.10

Notice that by the definition of the coordinates of the leftmost and rightmost particle one has $w = y - x$ and $u = x + y$ so that, as a consequence of (11) one obtains

$$\Pi^{(\theta)}((x,y),(x',y');\lambda) = \mathscr{G}^{(\theta)}((x+y, y-x),(x'+y', y'-x');\lambda)\ . \tag{95}$$

To obtain an explicit expression we use translation invariance

$$\mathscr{G}((u,w);(u',w');\lambda) = \mathscr{G}((0,w);(u'-u,w');\lambda) \tag{96}$$

and we rewrite $\mathscr{G}^{(\theta)}$ as follows

$$\mathscr{G}^{(\theta)}((u,w);(u',w');\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G^{(\theta)}(w,w',\kappa,\lambda)e^{i\kappa(u'-u)}\,d\kappa, \tag{97}$$

where on the right hand side we can insert the expression for $G^{(\theta)}$ that appears in Theorem 2.8. In doing so it is also useful to write an integral representation for the terms $\zeta_{\lambda,\kappa}^{|w'-w|}$ and $\zeta_{\lambda,\kappa}^{w'+w}$ in Theorem 2.8. This can be obtained by noticing that for any ζ with $|\zeta| < 1$ one has

$$\sum_{x=0}^{\infty}\zeta^x\cos\left(mx\right) = \frac{1-\zeta\cos(m)}{1+\zeta^2-2\zeta\cos(m)} \tag{98}$$

so that

$$\sum_{x\in\mathbb{Z}} \zeta^{|x|} e^{-imx} = \frac{1-\zeta^2}{1+\zeta^2-2\zeta\cos(m)} \; . \tag{99}$$

Thus we have

$$\zeta^{|x|} \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}\, \frac{\zeta^{-1}-\zeta}{\zeta^{-1}+\zeta-2\cos(m)}\, dm \; . \tag{100}$$

We are indeed allowed to use this expression since $|\zeta_{\lambda,\kappa}| \leq 1$, as it can immediately be seen from (30) observing that $x_{\lambda,\kappa} \geq 1$ for all $\alpha, \lambda \geq 0$. All in all, combining together Eq. (97), Theorem 2.8 and Eq. (100), one arrives to

$$\mathscr{G}^{(\theta)}((u,w);(u',w');\lambda) = \tag{101}$$

$$= \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{f^{(\theta)}_{\lambda,\kappa}(w,w')\, e^{i\kappa(u'-u)}}{\left(1+\frac{\lambda}{2}\right)-\nu_\kappa\cos(m)} \left(e^{im(w'-w)} + e^{im(w'+w)}\cdot\left(2\,\frac{\mathscr{Z}^{(0)}_{\lambda,\kappa}}{\mathscr{Z}^{(\theta)}_{\lambda,\kappa}}-1\right)\right) dm\, d\kappa \; .$$

This expression can now be used in Eq. (95). Defining

$$\kappa_1 = \kappa - m, \quad \kappa_2 = \kappa + m, \tag{102}$$

and considering the case $x \neq y$ one obtains (34). □

4 Scaling limits: proof of Theorem 2.11

This section is organised as follows: we first prove several results on the limiting process for the properly rescaled joint process of distance and sum coordinates, and then we use these computations to show that the candidate scaling limit is the "correct scaling limit".

Specifically, in the preliminary section 4.1 we compute the Fourier-Laplace transform of the probability density of the standard sticky Brownian motion. In section 4.2 we obtain the Fourier-Laplace transform of the transition density of the candidate scaling limit, i.e. a system of two Brownian motions with a sticky interaction defined via their local intersection time. Further, in section 4.3 we show that the Fourier-Laplace transform for a generic value of the stickiness parameter can be written as a convex combination of the Fourier-Laplace transforms of two limiting cases, i.e. absorbing and reflecting. With these results in our hand, we then continue by proving in section 4.4 convergence of the appropriately rescaled discrete two-particle process to the system of sticky Brownian motions (Theorem 2.11). The convergence in distribution is inferred from the convergence of the Fourier-Laplace transforms. Finally section 4.5 contains the proof of Corollary 2.14, that deals with the convergence of the expected local time.

4.1 Standard Brownian motion sticky at the origin

We start with a preliminary computation that involves just a single sticky Brownian motion (which is indeed the scaling limit of the distance process). We recall the definition of the sticky Brownian motion, see [12, 15] for more background on such process. For all $t \geq 0$, let $L(t)$ be the local time at the origin of a standard Brownian motion $B(t)$ and let $\gamma > 0$. Set

$$s^{-1}(t) = t + \gamma L(t). \tag{103}$$

The (one-sided) standard sticky Brownian motion $B^S(t)$ on $\mathbb{R}_+$ with sticky boundary at the origin and stickiness parameter $\gamma > 0$ is defined as the time changed standard reflected Brownian motion, i.e.

$$B^S(t) = |B(s(t))|. \tag{104}$$

Using the expression for the joint density of $(|B(t)|, L(t))$ (formula (3.14) in [15])

$$\mathbb{P}_0(|B(r)| \in dx, L(r) \in dy) = 2\frac{x+y}{\sqrt{2\pi r^3}} e^{-\frac{(x+y)^2}{2r}} dxdy \qquad x, y \geq 0, \tag{105}$$

one can compute the Fourier-Laplace transform of $B^S(t)$ that is defined as

$$\psi_0^S(m, \lambda, \gamma) := \int_0^\infty \mathbb{E}_0 \left[e^{-imB^S(t)} \right] e^{-\lambda t} dt, \tag{106}$$

where the subscript 0 denotes the initial position of $B^S(t)$.

Lemma 4.1 (Fourier-Laplace transform of standard sticky Brownian motion)**.** *We have*

$$\psi_0^S(m, \lambda, \gamma) = \frac{\sqrt{2}}{\sqrt{\lambda} + \frac{\gamma}{\sqrt{2}}\lambda} \int_0^\infty e^{-\sqrt{2\lambda}x - imx} dx + \frac{\frac{\gamma}{\sqrt{2}}}{\sqrt{\lambda} + \frac{\gamma}{\sqrt{2}}\lambda}. \tag{107}$$

Proof. We rewrite (106) using (104) and then apply the change of variable $s(t) = r$ to obtain

$$\begin{aligned} \psi_0^S(m, \lambda, \gamma) - \frac{1}{\lambda} &= \mathbb{E}_0 \left[\int_0^\infty \left(e^{-imB^S(t)} - 1 \right) e^{-\lambda t} dt \right] \\ &= \mathbb{E}_0 \left[\int_0^\infty \left(e^{-im|B(r)|} - 1 \right) e^{-\lambda(r+\gamma L(r))} (dr + \gamma dL(r)) \right]. \end{aligned} \tag{108}$$

The local time $L(t)$ only grows when $B(t)$ is at the origin implying that the term into the round bracket is zero in the integral with respect to $dL(r)$. As a consequence we have

$$\psi_0^S(m, \lambda, \gamma) - \frac{1}{\lambda} = \mathbb{E}_0 \left[\int_0^\infty \left(e^{-im|B(r)|} - 1 \right) e^{-\lambda(r+\gamma L(r))} dr \right]. \tag{109}$$

Then (107) follows by using the expression (105) and the formula for the Laplace transform

$$\int_0^\infty e^{-\lambda r} \frac{a}{\sqrt{2\pi r^3}} e^{-a^2/2r} dr = e^{-\sqrt{2\lambda}a} \qquad a > 0. \tag{110}$$

□

4.2 The joint sticky process

Let $\tilde{B}(t)$ and $B(t)$ be two independent Brownian motions starting, respectively, from 0 and from $z \geq 0$. Let $s(t)$ be defined via (103), with $L(t)$ being now the local time of $B(t)$. We compute in this section the Fourier-Laplace transform of the candidate scaling limit, i.e. the joint process $(\tilde{B}(2t-s(t)), |B(s(t))|)$, that is defined as

$$\Psi_z^S(\kappa, m, \lambda, \gamma) := \int_0^\infty \mathbb{E}_{0,z}\left[e^{-i\kappa\tilde{B}(2t-s(t))-im|B(s(t))|}\right] e^{-\lambda t} dt, \tag{111}$$

where the expectation $\mathbb{E}_{0,z}$ denotes expectation w.r.t. both the $\tilde{B}(t)$ Brownian motion that starts from 0 and the $B(t)$ process that starts from $z \geq 0$.

We start with the following Lemma that extends (105) to a positive initial condition.

Lemma 4.2 (Joint density of reflected Brownian motion and local time)**.** *For all $z > 0$ we have*

$$\begin{aligned} \mathbb{P}_z(|B(t)| \in dx, L(t) \in dy) &= \frac{1}{\sqrt{2\pi t}} \cdot \left(e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(z+x))^2}{2t}}\right) \delta_0(y)\, dx\, dy \\ &+ \left(2\int_0^t \frac{x+y}{\sqrt{2\pi(t-s)^3}}\, e^{-\frac{(x+y)^2}{2(t-s)}} \cdot \frac{z}{\sqrt{2\pi s^3}}\, e^{-\frac{z^2}{2s}}\, ds\right) dx\, dy \end{aligned} \tag{112}$$

Proof. Let $\nu_z(\cdot)$ be the probability density function of τ_z, the first hitting time of 0 for a Brownian motion starting from $z > 0$, i.e.

$$\nu_z(s) = \frac{z}{\sqrt{2\pi s^3}}\, e^{-\frac{z^2}{2s}}. \tag{113}$$

By conditioning to the time t being smaller or larger than τ_z we have

$$\begin{aligned} \mathbb{P}_z(|B(t)| \in dx, L(t) \in dy) &= \mathbb{P}_z(|B(t)| \in dx, \min_{s\leq t} B(s) > 0) \cdot \delta_0(y)\, dy \\ &+ \int_0^t \mathbb{P}_0(|B(t-s)| \in dx, L(t-s) \in dy)\, \nu_z(s) ds. \end{aligned} \tag{114}$$

The reflection principle for Brownian motion gives

$$\mathbb{P}_z\left(|B(t)| \in dx, \min_{s\leq t} B(s) > 0\right) = \frac{1}{\sqrt{2\pi t}} \cdot \left(e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(z+x)^2}{2t}}\right) dx,$$

whereas the use of (105) and (113) yields

$$\begin{aligned} &\int_0^t \mathbb{P}_0(|B(t-s)| \in dx, L(t-s) \in dy)\, \nu_z(s) ds \\ &= \left(2\int_0^t \frac{x+y}{\sqrt{2\pi(t-s)^3}}\, e^{-\frac{(x+y)^2}{2(t-s)}} \cdot \frac{z}{\sqrt{2\pi s^3}}\, e^{-\frac{z^2}{2s}}\, ds\right) dx\, dy. \end{aligned} \tag{115}$$

This concludes the proof. $\square$

Armed with the previous Lemma we can compute the Fourier-Laplace transform defined in (111).

Lemma 4.3 (Fourier-Laplace transform of the joint sticky process). *For all $z \geq 0$ we have*

$$\Psi_z^S(\kappa, m, \lambda, \gamma) = \frac{\gamma\, e^{-\sqrt{\kappa^2+2\lambda}z}}{\gamma(\kappa^2+\lambda)+\sqrt{\kappa^2+2\lambda}} +$$

$$\frac{1}{\sqrt{\kappa^2+2\lambda}}\left\{\frac{\sqrt{\kappa^2+2\lambda}-\gamma(\kappa^2+\lambda)}{\sqrt{\kappa^2+2\lambda}+\gamma(\kappa^2+\lambda)}\int_0^\infty e^{-imx}\, e^{-\sqrt{\kappa^2+2\lambda}|z+x|}\, dx + \int_0^\infty e^{-imx} e^{-\sqrt{\kappa^2+2\lambda}|z-x|}\, dx\right\}.$$

Proof. We follow a strategy similar to the one in the proof of Lemma 4.1. It is convenient to write

$$\Psi_z^S(\kappa, m, \lambda, \gamma) - f_z(\kappa, \lambda) = \int_0^\infty \mathbb{E}_{0,z}\left[e^{-i\kappa\tilde{B}(2t-s(t))}\left(e^{-imB(s(t))}-1\right)\right] e^{-\lambda t} dt,$$

where

$$f_z(\kappa, \lambda) = \int_0^\infty \mathbb{E}_{0,z}\left[e^{-i\kappa\tilde{B}(2t-s(t))}\right] e^{-\lambda t} dt.$$

and $\mathbb{E}_{0,z}$ denotes expectation with respect to the $\tilde{B}(t)$ process started at 0 and the $B(t)$ process started at z. We apply the change of variable $s(t) = r$ to obtain

$$\begin{aligned}\Psi_z^S(\kappa, m, \lambda, \gamma) - f_z(\kappa, \lambda) &= \mathbb{E}_{0,z}\left[\int_0^\infty e^{-i\kappa\tilde{B}(r+2\gamma L(r))}\left(e^{-im|B(r)|}-1\right) e^{-\lambda(r+\gamma L(r))}(dr+\gamma dL(r))\right] \\ &= \mathbb{E}_{0,z}\left[\int_0^\infty e^{-i\kappa\tilde{B}(r+2\gamma L(r))}\left(e^{-im|B(r)|}-1\right) e^{-\lambda(r+\gamma L(r))} dr\right], \qquad (116)\end{aligned}$$

where the last equality uses again that the local time $L(t)$ only grows when $B(t)$ is at the origin. Using then the independence of $\tilde{B}(t)$ and $B(t)$, and the expression for the characteristic function of the standard Brownian motion, we arrive to

$$\begin{aligned}\Psi_z^S(\kappa, m, \lambda, \gamma) &= \mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))}\, e^{-im|B(r)|}\, e^{-\lambda(r+\gamma L(r))} dr\right] \\ &+ f_z(\kappa, \lambda) - \mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))} e^{-\lambda(r+\gamma L(r))} dr\right] \qquad (117)\end{aligned}$$

We now evaluate separately the two terms on the r.h.s.. For the first term, thanks to the formula (112), we may write

$$\mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))}\, e^{-im|B(r)|}\, e^{-\lambda(r+\gamma L(r))} dr\right] =$$

$$\int_0^\infty e^{-imx}\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\frac{1}{\sqrt{2\pi r}}\left(e^{-\frac{(z-x)^2}{2r}} - e^{-\frac{(z+x)^2}{2r}}\right) dr\right] dx + \qquad (118)$$

$$\int_0^\infty e^{-imx}\int_0^\infty e^{-(\kappa^2+\lambda)\gamma y}\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\left(2\int_0^r \frac{x+y}{\sqrt{2\pi(r-s)^3}}\, e^{-\frac{(x+y)^2}{2(r-s)}} \cdot \frac{z}{\sqrt{2\pi s^3}}\, e^{-\frac{z^2}{2s}}\, ds\right) dr\right] dy\, dx. \qquad (119)$$

In (118) we may use the formula for the Laplace transform

$$\int_0^\infty e^{-\lambda r}\frac{1}{\sqrt{2\pi r}} e^{-a^2/2r} = \frac{e^{-a\sqrt{2\lambda}}}{\sqrt{2\lambda}}, \qquad (120)$$

and in (119) we may employ the Laplace transform (110) and the convolution rule. All in all, we find

$$
\begin{aligned}
&\mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))}\, e^{-im|B(r)|}\, e^{-\lambda(r+\gamma L(r))} dr\right] = \\
&\frac{1}{\sqrt{\kappa^2+2\lambda}}\int_0^\infty e^{-imx}\left(e^{-|z-x|\sqrt{\kappa^2+2\lambda}} - e^{-|z+x|\sqrt{\kappa^2+2\lambda}}\right) dx + \\
&\frac{2}{\gamma(\kappa^2+\lambda)+\sqrt{\kappa^2+2\lambda}}\int_0^\infty e^{-imx} e^{-\sqrt{\kappa^2+2\lambda}(x+z)}\, dx.
\end{aligned}
\tag{121}
$$

For the second term on the r.h.s. of (117) we observe that

$$
\begin{aligned}
f_z(\kappa,\lambda) &= \mathbb{E}_{0,z}\left[\int_0^\infty e^{-i\kappa\tilde{B}(2t-s(t))} e^{-\lambda t} dt\right] \\
&= \mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))} e^{-\lambda(r+\gamma L(r))} d(r+\gamma L(r))\right].
\end{aligned}
$$

As a consequence we find

$$
f_z(\kappa,\lambda) - \mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))} e^{-\lambda(r+\gamma L(r))} dr\right] = \gamma\mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))} e^{-\lambda(r+\gamma L(r))} dL(r)\right].
$$

This last integral can be evaluated integrating by parts, yielding

$$
\begin{aligned}
&\gamma\mathbb{E}_z\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\, e^{-\gamma(\kappa^2+\lambda)L(r)} dL(r)\right] \\
&= -\frac{1}{\kappa^2+\lambda}\,\mathbb{E}_z\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\, d\left(e^{-\gamma(\kappa^2+\lambda)L(r)}\right)\right] \\
&= \frac{1}{\kappa^2+\lambda}\left\{\mathbb{E}_z\left[\int_0^\infty d\left(e^{-(\frac{\kappa^2}{2}+\lambda)r}\right) e^{-\gamma(\kappa^2+\lambda)L(r)}\right] + \mathbb{E}_z\left[e^{-\gamma(\kappa^2+\lambda)L(0)}\right]\right\} \\
&= \frac{1}{\kappa^2+\lambda}\left\{-\left(\frac{\kappa^2}{2}+\lambda\right)\mathbb{E}_z\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\, e^{-\gamma(\kappa^2+\lambda)L(r)}\, dr\right] + 1\right\} \\
&= -\frac{\frac{\kappa^2}{2}+\lambda}{\kappa^2+\lambda}\,\mathbb{E}_z\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\, e^{-\gamma(\kappa^2+\lambda)L(r)}\, dr\right] + \frac{1}{\kappa^2+\lambda}.
\end{aligned}
$$

Furthermore, by using formula (112), we have

$$
\begin{aligned}
&\mathbb{E}_z\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\, e^{-\gamma(\kappa^2+\lambda)L(r)}\, dr\right] = \\
&\int_0^\infty\int_0^\infty \frac{1}{\sqrt{2\pi r}}\, e^{-(\frac{\kappa^2}{2}+\lambda)r}\cdot\left(e^{-\frac{(z-x)^2}{2r}} - e^{-\frac{(z+x)^2}{2r}}\right) dr\, dx + \\
&2\int_0^\infty\int_0^\infty e^{-\gamma(\kappa^2+\lambda)y}\left[\int_0^\infty e^{-(\frac{\kappa^2}{2}+\lambda)r}\left(\int_0^r \frac{x+y}{\sqrt{2\pi(r-s)^3}}\, e^{-\frac{(x+y)^2}{2(r-s)}}\, \frac{z}{\sqrt{2\pi s^3}}\, e^{-\frac{z^2}{2s}}\, ds\right) dr\right] dx\, dy = \\
&\frac{1}{\sqrt{\kappa^2+2\lambda}}\int_0^\infty \left(e^{-|z-x|\sqrt{\kappa^2+2\lambda}} - e^{-|z+x|\sqrt{\kappa^2+2\lambda}}\right) dx + \frac{2\, e^{-\sqrt{\kappa^2+2\lambda}z}}{\gamma(\kappa^2+\lambda)+\sqrt{\kappa^2+2\lambda}}\cdot\frac{1}{\sqrt{\kappa^2+2\lambda}},
\end{aligned}
$$

where in the last equality we used again the Laplace transforms (110) and (120). Summarizing, for the second term on the r.h.s. of (117) we have

$$
\begin{aligned}
&f_z(\kappa,\lambda) - \mathbb{E}_z\left[\int_0^\infty e^{-\frac{\kappa^2}{2}(r+2\gamma L(r))} e^{-\lambda(r+\gamma L(r))} dr\right] \\
&= -\frac{1}{2}\frac{\sqrt{\kappa^2+2\lambda}}{\kappa^2+\lambda}\int_0^\infty \left(e^{-|z-x|\sqrt{\kappa^2+2\lambda}} - e^{-|z+x|\sqrt{\kappa^2+2\lambda}}\right) dx \\
&-\frac{\sqrt{\kappa^2+2\lambda}}{\kappa^2+\lambda}\frac{e^{-\sqrt{\kappa^2+2\lambda}(z-1)}}{\gamma(\kappa^2+\lambda)+\sqrt{\kappa^2+2\lambda}} + \frac{1}{\kappa^2+\lambda}. \qquad (122)
\end{aligned}
$$

Hence, substituting (121) and (122) into (117), the statement of the Lemma follows after elementary simplifications. □

4.3 Limiting cases

In this section we show that the joint sticky process interpolates between two limiting cases.

4.3.1 Reflection: $\gamma = 0$

Let $B(t)$ be a Brownian motion starting from 0 and $B^R(t)$ be a Brownian motion on $\mathbb{R}^+$ reflected at 0 and starting from $z \geq 0$. Suppose they are independent and denote by Ψ_z^R the Laplace transform of the characteristic function of the joint process, i.e.

$$
\Psi_z^R(\kappa, m, \lambda) := \int_0^\infty \mathbb{E}_{0,z}\left[e^{-i\kappa\tilde{B}(t)-imB^R(t)}\right] e^{-\lambda t} dt. \qquad (123)
$$

Then we have

$$
\Psi_z^R(\kappa, m, \lambda) = \frac{1}{\sqrt{2\lambda+\kappa^2}}\int_0^\infty e^{-imx}\left(e^{-\sqrt{2\lambda+\kappa^2}|x+z|} + e^{-\sqrt{2\lambda+\kappa^2}|x-z|}\right) dx. \qquad (124)
$$

Indeed, from the knowledge of the transition probability of the joint process, an immediate computation gives

$$
\int_0^\infty \mathbb{E}_{0,z}\left[e^{-i(\kappa\tilde{B}(t)+mB^R(t))}\right] e^{-\lambda t} dt = \frac{1}{2\pi}\int_0^\infty e^{-imx}\int_{-\infty}^\infty \frac{e^{i\bar{m}(x+z)} + e^{i\bar{m}(x-z)}}{\lambda + \frac{\kappa^2}{2} + \frac{\bar{m}^2}{2}} d\bar{m}\, dx.
$$

This yields (124) using the Fourier transform

$$
\int_{-\infty}^\infty \frac{e^{ika}}{k^2+b^2} dk = \frac{\pi}{b} e^{-b|a|}. \qquad (125)
$$

.

4.3.2 Absorption: $\gamma \to \infty$

Let $\tilde{B}(t)$ be a standard Brownian motion and let $B^A(t)$ be an independent Brownian motion on $\mathbb{R}^+$ absorbed in 0 and starting from $z > 0$. Define τ_z as the absorption time, i.e. $\tau_z = \inf\{t \geq 0 : B^A(t) = 0\}$.

The process $\tilde{B}(2t - t \wedge \tau_z)$ describes the evolution of the sum of two coalescing Brownian motions started at two positions such that initially the sum is zero. Indeed, given the coalescing time τ_z (i.e. the hitting time of level zero for the distance of the two coalescing Brownian motions when the distance is initially $z > 0$), the center of mass of the two coalescing Brownian motion evolves as the sum of two independent standard Brownians until the coalescing time and, after coalescence, it evolves as a Brownian started at the position where coalescence occurred with double speed. Define

$$\Psi_z^A(\kappa, m, \lambda) = \int_0^\infty \mathbb{E}_{0,z}\left[e^{-i\kappa\tilde{B}(2t - t\wedge\tau_z) - imB^A(t)}\right] e^{-\lambda t} dt, \tag{126}$$

then we have

$$\Psi_z^A(\kappa, m, \lambda) = \frac{1}{\sqrt{2\lambda + \kappa^2}} \int_0^\infty e^{-imx'}\left(e^{-\sqrt{2\lambda+\kappa^2}|x-z|} - e^{-\sqrt{2\lambda+\kappa^2}|x+z|}\right) dx + \frac{e^{-z\sqrt{2\lambda+\kappa^2}}}{\lambda + \kappa^2}. \tag{127}$$

To show this one starts from

$$\Psi_z^A(\kappa, m, \lambda) = \int_0^\infty \mathbb{E}_{0,z}\left[e^{-i\kappa\tilde{B}(t) - imB^A(t)}\mathbf{1}_{t<\tau_z}\right] e^{-\lambda t} dt + \int_0^\infty \mathbb{E}_0\left[e^{-i\kappa\tilde{B}(2t-\tau_z)}\mathbf{1}_{t\geq\tau_z}\right] e^{-\lambda t} dt. \tag{128}$$

The first term in the r.h.s. of (128) is given by

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty e^{-imx}\int_{-\infty}^\infty \frac{e^{i\bar{m}(x-z)} - e^{i\bar{m}(x+z)}}{\lambda + \frac{\kappa^2}{2} + \frac{\bar{m}^2}{2}}\, d\bar{m}\, dx \\
&= \int_0^\infty e^{-imx}\,\frac{1}{\sqrt{2\lambda+\kappa^2}}\left(e^{-\sqrt{2\lambda+\kappa^2}|x-z|} - e^{-\sqrt{2\lambda+\kappa^2}|x+z|}\right) dx.
\end{aligned}$$

For the second term in the r.h.s of (128), let $\nu_z(\cdot)$ be the probability density of τ_z (cf. (113)). Then, using the fact that

$$\int_0^\infty e^{-\lambda t}\,\nu_z(t)\, dt = e^{-z\sqrt{2\lambda}}, \tag{129}$$

we obtain that the second term in the r.h.s of (128) is equal to

$$\int_0^\infty dt\, e^{-\lambda t}\int_0^t ds\,\nu_z(s)\int_{-\infty}^\infty dy'\, e^{-i\kappa y'}\int_{-\infty}^\infty dy\,\frac{e^{-\frac{y^2}{2s}}}{\sqrt{2\pi s}}\,\frac{e^{\frac{-(y'-y)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} = \frac{e^{-z\sqrt{2\lambda+\kappa^2}}}{\lambda+\kappa^2} \tag{130}$$

This is obtained by first doing the integrals in y and y' as Fourier transforms of suitable Brownian kernels and then by applying integration by parts to the dt integral followed by the use of formula (129).

4.3.3 Summary

Notice that we can rewrite the function Ψ_z^S as an interpolation between Ψ_z^R and Ψ_z^A as follows

$$\Psi_z^S(\kappa,m,\lambda,\gamma) = c^{(\gamma)}(\kappa,\lambda)\ \Psi_z^R(\kappa,m,\lambda) + \left(1 - c^{(\gamma)}(\kappa,\lambda)\right)\ \Psi_z^A(\kappa,m,\lambda) \tag{131}$$

with

$$c^{(\gamma)}(\kappa,\lambda) = \frac{\sqrt{\kappa^2+2\lambda}}{\sqrt{\kappa^2+2\lambda}+\gamma\,(\kappa^2+\lambda)} \tag{132}$$

Notice also that

$$\Psi_z^S(\kappa,m,\lambda,0) = \ \Psi_z^R(\kappa,m,\lambda) \qquad \text{and} \qquad \lim_{\gamma\to+\infty}\Psi_z^S(\kappa,m,\lambda,\gamma) = \ \Psi_z^A(\kappa,m,\lambda) \tag{133}$$

since $c^{(0)}(\kappa,\lambda) = 1$ and $\lim_{\gamma\to\infty} c^{(\gamma)}(\kappa,\lambda) = 0$.

4.4 Scaling limit

Proposition 4.4 (Convergence of Fourier-Laplace transform)**.** *For all $\kappa, m \in \mathbb{R}, \lambda > 0$ we have*

$$\lim_{\epsilon\to 0}\int_0^\infty \mathbb{E}_{u,w}\left[e^{-i(\kappa(U_\epsilon(t)-U)+mW_\epsilon(t))}\right] e^{-\lambda t}\,dt = \begin{cases} \Psi_W^R(\kappa,m,\lambda) & \textit{in the Reflected Regime,} \\ \\ \Psi_W^S(\kappa,m,\lambda,\gamma) & \textit{in the Sticky Regime,} \\ \\ \Psi_W^A(\kappa,m,\lambda) & \textit{in the Absorbed Regime.} \end{cases} \tag{134}$$

Proof. Using (97) and (101) we can rewrite the Fourier-Laplace transform of Theorem 2.8 in integral form:

$$G^{(\theta)}(w,w',\kappa,\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f^{\theta}_{\lambda,\kappa}(w,w')}{2+\lambda-2\nu_\kappa\cos(\bar m)}\left(e^{i\bar m(w'-w)} + e^{i\bar m(w'+w)}\left(2\,\frac{\mathscr{Z}^{(0)}_{\lambda,\kappa}}{\mathscr{Z}^{(\theta)}_{\lambda,\kappa}} - 1\right)\right)d\bar m.$$

Furthermore, by taking the Fourier transform with respect to the w variable, one finds

$$\begin{aligned}
&\int_0^\infty \mathbb{E}_{u,w}\left[e^{-i\kappa(u(t)-u)-imw(t)}\right] e^{-\lambda t}\,dt \qquad (135)\\
&= \sum_{w'\geq 0} e^{-imw'} G^{(\theta)}(w,w',\kappa,\lambda)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sum_{w'\geq 0} f^{(\theta)}_{\lambda,\kappa}(w,w')\,e^{i(\bar m-m)w'}}{2+\lambda-2\nu_\kappa\cos(\bar m)}\left(e^{-i\bar m w} + e^{i\bar m w}\left(2\,\frac{\mathscr{Z}^{(0)}_{\lambda,\kappa}}{\mathscr{Z}^{(\theta)}_{\lambda,\kappa}} - 1\right)\right)d\bar m.
\end{aligned}$$

Hence, for the full Fourier-Laplace transform of the scaled process, one has

$$\begin{aligned}
&\int_0^\infty \mathbb{E}_{u,w}\left[e^{-i\sqrt{2}(\kappa(U_\epsilon(t)-\frac{\epsilon u}{\sqrt{2}})+mW_\epsilon(t))}\right] e^{-\lambda t}\,dt = \qquad (136)\\
&= \frac{\epsilon^2}{2\pi}\int_{-\pi/\epsilon}^{\pi/\epsilon}\frac{\epsilon\sum_{w'\geq 0} f^{(\theta)}_{\epsilon^2\lambda,\epsilon\kappa}(w,w')\,e^{i\epsilon(\bar m-m)w'}}{2+\epsilon^2\lambda-2\nu_{\epsilon\kappa}\cos(\epsilon\bar m)}\left(e^{-iw\epsilon\bar m} + e^{iw\epsilon\bar m}\left(2\,\frac{\mathscr{Z}^{(0)}_{\epsilon^2\lambda,\epsilon\kappa}}{\mathscr{Z}^{(\theta)}_{\epsilon^2\lambda,\epsilon\kappa}} - 1\right)\right)d\bar m.
\end{aligned}$$

The explicit form of $f^{\theta}_{\lambda,\kappa}(w,w')$ in (28) gives

$$\epsilon \sum_{w'\geq 0} f^{(\theta)}_{\epsilon^2\lambda,\epsilon\kappa}(w,w')\, e^{i\epsilon(\bar{m}-m)w'} = \tag{137}$$
$$\epsilon \sum_{w'\geq 0} e^{i\epsilon(\bar{m}-m)w'} + \epsilon\, \frac{\theta}{2}\left(1 + \left(\nu^{-1}_{\epsilon\kappa}\zeta_{\epsilon^2\lambda,\epsilon\kappa} - 1\right)\mathbf{1}_{w=0}\right) - \frac{\epsilon}{2}.$$

Since $\nu^{-1}_{\epsilon\kappa}\zeta_{\epsilon^2\lambda,\epsilon\kappa} = 1 + o(\epsilon)$, recalling the definition of the parameter γ in (40), one has

$$\lim_{\epsilon\to 0} \epsilon \sum_{w'\geq 0} f^{(\theta)}_{\epsilon^2\lambda,\epsilon\kappa}(w,w')\, e^{i\epsilon(\bar{m}-m)w'} = \int_0^\infty e^{i(\bar{m}-m)x'}\, dx' + \frac{\gamma}{\sqrt{2}}. \tag{138}$$

Similarly one finds

$$\lim_{\epsilon\to 0} \frac{\mathscr{Z}^{(0)}_{\epsilon^2\lambda,\epsilon\kappa}}{\mathscr{Z}^{(\theta)}_{\epsilon^2\lambda,\epsilon\kappa}} = \frac{\sqrt{\kappa^2+\lambda}}{\sqrt{\kappa^2+\lambda} + \gamma\sqrt{2}\left(\kappa^2 + \frac{\lambda}{2}\right)}. \tag{139}$$

Hence, taking the limit $\epsilon \to 0$ in (136) and using (138) and (139) one has

$$\lim_{\epsilon\to 0}\int_0^\infty \mathbb{E}_{u,w}\left[e^{-i\sqrt{2}(\kappa(U_\epsilon(t)-\frac{\epsilon u}{\sqrt{2}})+mW_\epsilon(t))}\right] e^{-\lambda t}\, dt \tag{140}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\int_0^\infty e^{i(\bar{m}-m)x'}\, dx' + \frac{\gamma}{\sqrt{2}}}{\lambda+\kappa^2+\bar{m}^2}\left(e^{-iW\bar{m}} + e^{iW\bar{m}}\left(\frac{\sqrt{\kappa^2+\lambda} - \gamma\sqrt{2}\left(\kappa^2+\frac{\lambda}{2}\right)}{\sqrt{\kappa^2+\lambda}+\gamma\sqrt{2}\left(\kappa^2+\frac{\lambda}{2}\right)}\right)\right) d\bar{m}.$$

The previous expression can further simplified using the Fourier transform (125). In the end one arrives to:

$$\lim_{\epsilon\to 0}\int_0^\infty \mathbb{E}_{u,w}\left[e^{-i\sqrt{2}(\kappa(U_\epsilon(t)-U)+mW_\epsilon(t))}\right] e^{-\lambda t}\, dt = \tag{141}$$
$$c^{(\gamma)}(\sqrt{2}\kappa,\lambda)\psi^R_W(\sqrt{2}\kappa,\sqrt{2}m,\lambda) + \left(1 - c^{(\gamma)}(\sqrt{2}\kappa,\lambda)\right)\Psi^A_W(\sqrt{2}\kappa,\sqrt{2}m,\lambda)$$

from which (134) follows. $\square$

Proof of Theorem 2.11. First note that proposition 4.4 shows the convergence of the Fourier-Laplace transform of the transition probabilities. So the only left to prove is that we can get rid of the Laplace transform and have convergence in the time parameter, instead of the Laplace parameter. This is possible because convergence of resolvents implies convergence of semigroups. More precisely, denote by $T_\epsilon(t)$ the semigroup of the process $(U_\epsilon(t)-U, W_\epsilon(t))$, and let $T(t)$ be the semigroup of the claimed limiting process $(U(t), W(t))$, and A_ϵ, A the corresponding generators. By (134), we conclude that for compactly supported smooth functions $f : \mathbb{R}^2 \to \mathbb{R}$, the resolvents converge, i.e., for all $\lambda > 0$

$$\lim_{\epsilon\to 0}(\lambda - A_\epsilon)^{-1} f = (\lambda - A)^{-1} f.$$

Therefore, as smooth functions form a core for all A_ϵ as well as for A by [2], Theorem 2.2, we conclude also convergence of the semigroups, i.e., for all compactly supported continuous functions we have

$$\lim_{\epsilon\to 0} T_\epsilon(t)f = T(t)f,$$

which in turn implies the convergence of the processes in the sense of finite dimensional distributions.

□

4.5 Local time at 0

Lemma 4.5 (Laplace transform of probability to be at zero of sticky Brownian motion)**.** *Let $z \geq 0$ and let $\mathbb{P}_z(B^S(t) = 0)$ be the probability for a sticky Brownian motion started at z to be at 0 at time t. We have*

$$\int_0^\infty e^{-\lambda t}\, \mathbb{P}_z(B^S(t) = 0)\, dt = \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda}\, e^{-\sqrt{2\lambda} z}. \tag{142}$$

Proof. The l.h.s. of (142) can be rewritten as

$$\lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \int_0^\infty e^{-\lambda t}\, \mathbb{E}_z[e^{-im|B(s(t))|}]\, dt\, dm = \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \Psi_z^S(0, m, \lambda, \gamma)\, dm \tag{143}$$

Thus, using formula (131), we have that

$$\begin{aligned} \int_0^\infty e^{-\lambda t}\, \mathbb{P}_z(B^S(t) = 0)\, dt &= c^{(\gamma)}(0,\lambda) \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \Psi_z^R(0, m, \lambda)\, dm \\ &+ (1 - c^{(\gamma)}(0,\lambda)) \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \Psi_z^A(0, m, \lambda)\, dm. \end{aligned}$$

It is easy to see that

$$\lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \Psi_z^R(0, m, \lambda)\, dm = 0, \tag{144}$$

and

$$(1 - c^{(\gamma)}(0,\lambda)) \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \Psi_z^A(0, m, \lambda)\, dm = \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda}\, e^{-\sqrt{2\lambda} z}. \tag{145}$$

□

Proof of Proposition 2.14. The first statement (equations (44) and (45)) follows from the fact that

$$\int_0^\infty e^{-\lambda t}\, \mathbb{P}_w\,(w(t) = 0)\, dt = G^{(\theta)}(w, 0, 0, \lambda) \tag{146}$$

and the r.h.s. can be explicitly written thanks to Theorem 2.8. Furthermore, the diffusive scaling gives

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\,\mathbb{P}_w\left(W_\epsilon(t)=0\right)dt &= \epsilon^2 G^{(\theta)}(w,0,0,\lambda\epsilon^2)\\ &= \epsilon^2\zeta^w_{\lambda\epsilon^2}\,\frac{1+\sqrt{2}\,\gamma\epsilon^{-1}\zeta^{\mathbf{1}_{w=0}}_{\lambda\epsilon^2}}{\zeta^{-1}_{\lambda\epsilon^2}+(\sqrt{2}\,\gamma\lambda\epsilon-1)\zeta_{\lambda\epsilon^2}}\\ &= \frac{1+\sqrt{2}\,\gamma\epsilon^{-1}\left(1-\mathbf{1}_{w=0}\epsilon\sqrt{\lambda}\right)}{1+\epsilon\sqrt{\lambda}+(\sqrt{2}\,\gamma\lambda\epsilon-1)(1-\epsilon\sqrt{\lambda})}\,\epsilon^2\left(1-\epsilon\sqrt{\lambda}\right)^{\sqrt{2}W\epsilon^{-1}}\cdot(1+o(1))\end{aligned}$$

from which formula (46) follows. □

5 Applications in processes with duality.

5.1 Time dependent covariances

In this section we look at the (time dependent) covariance of $\eta_x(t)$ and $\eta_y(t)$ for the reference process with generator (16) when initially started from a product measure. We recall form section 2.4 that the reference process encompasses the generalized symmetric exclusion process (SEP(j)), the symmetric inclusion process (SIP(k)) and the independent random walk process (IRW). We will denote by ν the initial product measure and by

$$\rho(x):=\int\eta_x d\nu \qquad \text{and} \qquad \chi(x):=\int\eta_x(\eta_x-1)d\nu \tag{147}$$

We further denote

$$\rho_t(x)=\sum_y p_t(x,y)\rho(y)=\int\mathbb{E}_\eta(\eta_x(t))d\nu \tag{148}$$

We will denote by X_t,Y_t, resp. $\widetilde{X}_t,\widetilde{Y}_t$ the positions of two dual particles, resp. two independent particles, and by $\mathbb{E}_{x,y}$, the corresponding expectations when particles start from x,y. The following proposition describes time-dependent covariances of particle numbers at time $t>0$ when starting from an arbitrary initial distribution ν, in terms of two dual particles.

Proposition 5.1 (Time dependent covariances throught dual particles)**.** *Let $\{\eta(t):t\geq 0\}$ be a self-dual process with generator* (16) *and $\alpha=1$. Then*

$$\begin{aligned}\Xi^{(\theta)}(t,x,y;\nu) &= (1+\theta\delta_{x,y})\Big\{\mathbb{E}_{x,y}\left[\rho(X_t)\rho(Y_t)-\rho(\widetilde{X}_t)\rho(\widetilde{Y}_t)\right]\\ &+ \mathbb{E}_{x,y}\left[\mathbf{1}_{X_t=Y_t}\left(\frac{1}{1+\theta}\,\chi(X_t)-\rho(X_t)^2\right)\right]\Big\}\\ &+ \delta_{x,y}\left(\theta\rho_t(x)^2+\rho_t(x)\right)\end{aligned}\tag{149}$$

where

$$\theta=\begin{cases}0 & \mathit{IRW}\\ +\frac{1}{k} & \mathit{SIP}(k)\\ -\frac{1}{j} & \mathit{SEP}(j).\end{cases}\tag{150}$$

Proof. To prove the theorem we use duality relations. From section 2.4, duality functions for one and two particles dual configurations are given by:

$$D(\delta_x, \eta) = c_1 \eta_x, \tag{151}$$

$$D(\delta_x + \delta_y, \eta) = \begin{cases} c_1^2 \eta_x \eta_y & \text{for } x \neq y \\ c_2 \eta_x (\eta_x - 1) & \text{for } x = y, \end{cases} \tag{152}$$

with

$$c_1 := \begin{cases} 1 & \text{IRW} \\ \frac{1}{k} & \text{SIP}(k) \\ \frac{1}{j} & \text{SEP}(j) \end{cases} \qquad c_2 := \begin{cases} 1 & \text{IRW} \\ \frac{1}{k(k+1)} & \text{SIP}(k) \\ \frac{1}{j(j-1)} & \text{SEP}(j). \end{cases} \tag{153}$$

Hence, for all cases, $\theta + 1 = c_1^2 / c_2$. Then we have

$$\rho(x) = \frac{1}{c_1} \int D(\delta_x, \eta) d\nu \qquad \text{and} \qquad \chi(x) = \frac{1}{c_2} \int D(2\delta_x, \eta) d\nu \tag{154}$$

We denote by $p_t(x, y)$ the transition probability for one dual particle to go from x to y in time t. Moreover we denote by $p_t(x, y; u, v)$ the transition probability for two dual particles to go from x, y to u, v in time t. We consider two cases: the first being $x \neq y$. Using self-duality, we write

$$\begin{aligned}
& \mathbb{E}_\eta [\eta_x(t)\eta_y(t)] - \rho_t(x)\mathbb{E}_\eta [\eta_y(t)] - \rho_t(y)\mathbb{E}_\eta [\eta_x(t)] + \rho_t(x)\rho_t(y) \\
= \; & \frac{1}{c_1^2} \mathbb{E}_\eta [D(\delta_x + \delta_y, \eta(t))] - \rho_t(x) \frac{1}{c_1} \mathbb{E}_\eta [D(\delta_y, \eta(t))] \\
- \; & \rho_t(y) \frac{1}{c_1} \mathbb{E}_\eta [D(\delta_x, \eta(t))] + \rho_t(x)\rho_t(y) \\
= \; & \sum_{u \neq v} p_t(x, y; u, v) \eta_u \eta_v + \frac{c_2}{c_1^2} \sum_u p_t(x, y; u, u) \eta_u (\eta_u - 1) \\
- \; & \rho_t(x) \sum_v p_t(y, v) \eta_v - \rho_t(y) \sum_u p_t(x, u) \eta_u + \rho_t(x)\rho_t(y).
\end{aligned} \tag{155}$$

We now integrate the η-variable over ν and obtain

$$\begin{aligned}
\Xi^{(\theta)}(t, x, y; \nu) \; = \; & \sum_{u,v} p_t(x, y; u, v) \rho(u) \rho(v) + \sum_u p_t(x, y; u, u) \left(\frac{1}{1+\theta} \, \chi(u) - \rho(u)^2 \right) \\
- \; & \rho_t(x)\rho_t(y) - \rho_t(y)\rho_t(x) + \rho_t(x)\rho_t(y) \\
= \; & \sum_{u,v} \left[p_t(x, y; u, v) - p_t(x, u) p_t(y, v) \right] \rho(u) \rho(v) \\
+ \; & \sum_u p_t(x, y; u, u) \left(\frac{1}{1+\theta} \, \chi(u) - \rho(u)^2 \right).
\end{aligned} \tag{156}$$

Now we turn to the second case $x = y$. We have

$$\begin{aligned}
\mathbb{E}_\eta\left[\eta_x^2(t)\right] &= \frac{1}{c_2}\,\mathbb{E}_\eta\left[D(2\delta_x,\eta(t))\right] + \frac{1}{c_1}\mathbb{E}_\eta\left[D(\delta_x,\eta(t))\right] \\
&= \frac{1}{c_2}\,\mathbb{E}_{2\delta_x}\left[D(\delta_{x(t)}+\delta_{y(t)},\eta)\right] + \frac{1}{c_1}\mathbb{E}_{\delta_x}\left[D(\delta_{x(t)},\eta)\right] \\
&= \frac{1}{c_2}\left(c_1^2\sum_{u\neq v} p_t(x,x;u,v)\eta_u\eta_v + c_2\sum_u p_t(x,x;u,u)\eta_u(\eta_u-1)\right) + \sum_u p_t(x,u)\eta_u.
\end{aligned}$$

Then

$$\int \mathbb{E}_\eta\left[\eta_x^2(t)\right] d\nu = (1+\theta)\sum_{u\neq v} p_t(x,x;u,v)\rho(u)\rho(v) + \sum_u p_t(x,x;u,u)\chi(u) + \sum_u p_t(x,u)\rho(u). \tag{157}$$

This leads to

$$\begin{aligned}
\Xi^{(\theta)}(t,x,x;\nu) &= (1+\theta)\sum_{u,v}\left[p_t(x,x;u,v) - p_t(x,u)p_t(x,v)\right]\rho(u)\rho(v) \\
&+ \sum_u p_t(x,x;u,u)\left(\chi(u) - (1+\theta)\rho(u)^2\right) \\
&+ \theta\rho_t(x)^2 + \rho_t(x).
\end{aligned}\tag{158}$$

This completes the proof of the Proposition. $\square$

When the initial measure ν is assumed to be an homogeneous product measure then the expression of the time dependent covariances via dual particles further simplifies. This is the content of the next proposition.

Proposition 5.2 (Case of homogeneous ν). *Suppose that ν is a homogeneous product measure then, for self-dual processes with generator* (16) *and $\alpha = 1$ we have*

$$\int_0^\infty e^{-\lambda t}\,\Xi^{(\theta)}(t,x,y;\nu)\,dt = (1+\theta\delta_{x,y})\left(\frac{\chi}{\theta+1} - \rho^2\right)\frac{\left(1+\theta\zeta_\lambda^{\mathbf{1}_{x=y}}\right)\zeta_\lambda^{\sqrt{2}|x-y|}}{\zeta_\lambda^{-1} + (\theta\lambda-1)\zeta_\lambda} + \frac{\delta_{x,y}}{\lambda}\left(\theta\rho_\nu^2 + \rho_\nu\right)$$

with ζ_λ as in (45).

Proof. We see from (149) that if ν is an homogeneous product measure then $\rho(X_t) = \rho$. As a consequence we have

$$\Xi^{(\theta)}(t,x,y;\nu) = (1+\theta\delta_{x,y})\left(\frac{\chi}{(1+\theta)} - \rho^2\right)\mathbb{P}_{x,y}\left(X_t = Y_t\right) + \delta_{x,y}\left(\theta\rho^2 + \rho\right). \tag{159}$$

Taking the Laplace transform and using (44) the result follows. $\square$

Remark 5.3. *Notice that if ν is an homogeneous product measure that satisfies the condition*

$$\int \eta_0(\eta_0 - 1)\, d\nu = (1+\theta)\left(\int \eta_0\, d\nu\right)^2 \tag{160}$$

then $\Xi^{(\theta)}(t,x,y;\nu)$ is not depending on t and more precisely,

$$\Xi^{(\theta)}(t,x,y;\nu) = 0 \qquad \textit{for} \quad x \neq y \qquad \textit{and} \qquad \Xi^{(\theta)}(t,x,x;\nu) = \chi + \rho - \rho^2.$$

This corresponds to the case where ν is a stationary product measure for which the covariance is constantly zero and the variance is equal at all times to the initial value, that is indeed given by

$$Var_\nu(\eta_0) = \int \eta_0^2\ d\nu - \rho^2 = \chi + \rho - \rho^2.$$

5.2 Scaling of variance and covariances in the sticky regime

Proof of Theorem 2.16. From Proposition (5.2) we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\, \Xi^{(\theta_\epsilon)}(\epsilon^{-a}t, \epsilon^{-1}x, \epsilon^{-1}y;\nu)\, dt \\
&= \epsilon^a \int_0^\infty e^{-\lambda\epsilon^a s}\, \Xi^{(\theta_\epsilon)}(s, \epsilon^{-1}x, \epsilon^{-1}y;\nu)\, ds \\
&= \epsilon^a \left(1+\sqrt{2}\gamma\epsilon^{-1}\delta_{x,y}\right)\left(\frac{\chi}{\sqrt{2}\gamma\epsilon^{-1}+1} - \rho^2\right) \frac{\left(1+\sqrt{2}\gamma\epsilon^{-1}\zeta_{\epsilon^a\lambda}^{\mathbf{1}_{x=y}}\right)\zeta_{\epsilon^a\lambda}^{\sqrt{2}|x-y|}}{\zeta_{\epsilon^a\lambda}^{-1} + (\sqrt{2}\gamma\epsilon^{a-1}\lambda - 1)\zeta_{\epsilon^a\lambda}} \\
&+ \frac{\delta_{x,y}}{\lambda}\left(\epsilon^{-1}\sqrt{2}\gamma\rho^2 + \rho\right)
\end{aligned}$$

Now we use the fact that $\zeta_\delta = (1-\sqrt{\delta})(1+o(1))$ for small δ and we obtain that, for $x \neq y$,

$$\int_0^\infty e^{-\lambda t}\, \Xi^{(\theta_\epsilon)}(\epsilon^{-a}t, \epsilon^{-1}x, \epsilon^{-1}y;\nu)\, dt = -\rho^2\sqrt{2}\gamma\ \frac{(1-\sqrt{\epsilon^a\lambda})^{\sqrt{2}|x-y|}}{2\sqrt{\lambda}\,\epsilon^{-(\frac{a}{2}-1)} + \sqrt{2}\gamma\lambda} \cdot (1+o(1))$$

as $\epsilon \to 0$. This produces the result for the covariance. For $x = y$, we get

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\, \Xi^{(\theta_\epsilon)}(\epsilon^{-a}t, \epsilon^{-1}x, \epsilon^{-1}x,\nu)\, dt = \\
&= \left\{\left(\chi - \left(1+\sqrt{2}\gamma\epsilon^{-1}\right)\rho^2\right) \frac{\left(\epsilon + \sqrt{2}\gamma(1-\sqrt{\epsilon^a\lambda})\right)}{2\sqrt{\lambda}\,\epsilon^{-(\frac{a}{2}-1)} + \sqrt{2}\gamma\lambda} + \frac{1}{\lambda}\left(\sqrt{2}\gamma\epsilon^{-1}\rho^2 + \rho\right)\right\} \cdot (1+o(1))
\end{aligned}$$

as $\epsilon \to 0$, from which the statement for the variance follows. □

5.3 Variance of the density fluctuation field

Proof of Theorem 2.18. From the definitions of the variance of the density fluctuation field (Eq. (51)) and of the time dependent covariances of the occupation numbers (Eq. (47)), we have

$$\mathbb{E}_\nu\left[\,(\mathscr{X}_\epsilon(\Phi,\eta,t))^2\,\right] = \epsilon^2 \sum_{x,y} \Phi(\epsilon x)\Phi(\epsilon y)\, \Xi^{(\theta_\epsilon)}(\epsilon^{-2}t, x, y;\nu). \tag{161}$$

Using Theorem 2.16 for the time dependent covariances we get

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t}\, \mathbb{E}_\nu \big[\, (\mathscr{X}_\epsilon(\Phi,\eta,t))^2 \, \big] \, dt \\
&= \epsilon^2 \sum_{x\neq y} \Phi(\epsilon x)\Phi(\epsilon y) \int_0^\infty e^{-\lambda t} \Xi^{(\theta_\epsilon)}(\epsilon^{-2}t,x,y;\nu)\, dt + \epsilon^2 \sum_x \Phi(\epsilon x)^2 \int_0^\infty e^{-\lambda t} \Xi^{(\theta_\epsilon)}(\epsilon^{-2}t,x,x;\nu)\, dt \\
&= \epsilon^2 \sum_{x\neq y} \Phi(\epsilon x)\Phi(\epsilon y) \frac{\gamma\rho^2\, e^{-\sqrt{\lambda}\epsilon|x-y|}}{\sqrt{2\lambda}+\gamma\lambda}\, (1+o(1)) + \epsilon \sum_x \Phi(\epsilon x)^2 \frac{2\sqrt{2}\gamma\rho^2}{2\lambda+\gamma\lambda\sqrt{2\lambda}}\, (1+o(1))
\end{aligned}
$$

from which the result follows. □

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