Consistent particle systems and duality

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Abstract

We consider consistent particle systems, which include independent random walkers, the symmetric exclusion and inclusion processes, as well as the dual of the KMP model. Consistent systems are such that the distribution obtained by first evolving n particles and then removing a particle at random is the same as the one given by a random removal of a particle at the initial time followed by evolution of the remaining $n-1$ particles.

We show that, for reversible systems, the property of consistency is equivalent to self-duality, thus obtaining a novel probabilistic interpretation of the self-duality property.

We also show that when a particle system is consistent, adding independent absorbing sites preserves this property. As a consequence, for a consistent system with absorption, the particle absorption probabilities satisfy universal recurrence relations. Since particle systems with absorption are often dual to boundary-driven non-equilibrium systems, the consistency property implies recurrence relations for expectations of duality polynomials in non-equilibrium steady states. We illustrate these relations with several examples.

1 Introduction

Particle systems with a duality property [9, 1] are very useful in the study of non-equilibrium steady states. This is because whenever we couple such systems to appropriately chosen reservoirs, they are dual to particle systems where the driving reservoirs are replaced by absorbing boundaries. As a consequence, the computation of correlation functions of the non-equilibrium steady state can be reduced to the computation of absorption probabilities of dual particles. These absorption probabilities can usually be obtained explicitly (in closed form) for a single particle, and in some exceptional cases, such as the symmetric simple exclusion process on a chain [3], also for many particles. In this paper, we focus on the property of consistency of particle systems and show that this property provides substantial additional information.

Consistent particle systems are informally defined as those that satisfy the following property: if we marginalize on the first $n-1$ coordinates the process $\{X^{(n)}(t) : t \geq 0\}$ describing the positions of n particles, we exactly obtain (in distribution) the process $\{X^{(n-1)}(t) :\geq 0\}$ describing the positions of $n-1$ particles. We require this property to hold for every number of particles, thus we are actually demanding consistency for a family of processes, indexed by the number of particles n. This property holds trivially for independent particles, here we are interested in characterizing the interacting particle systems for which it holds. The consistency property has already shown to be of great relevance in the context of the KMP model (see [9]), where local equilibrium was shown using consistency.

In this paper, we show that under the additional hypothesis of permutation invariance, consistency is equivalent with commutation with the so-called annihilation operator, which, in the presence of a reversible measure, is in turn equivalent with self-duality. We also show that adding absorbing sites conserves the consistency property. This leads to a set of recursion relations for the absorption probabilities. We apply these recursion relations to show universal properties of the non-equilibrium steady states of systems such as the exclusion and inclusion process coupled to boundary reservoirs.

The rest of the paper is organized as follows. In section 2 we introduce the notion of consistent permutation invariant particle systems and show that this property is equivalent with a commutation property between the generator of the configuration process and the annihilation operator. In section 3, we show the relation between consistency and (self-)duality and also show that consistency is preserved by adding sites where particles can be absorbed (independently for different particles). In section 4 we derive recursion relations for consistent systems with absorbing states on a chain. In section 5 we prove recursion relations correlations functions of non-equilibrium steady states and illustrate them in several examples.

2 Consistency and the annihilation operator

2.1 Basic definitions

We consider a system of n particles moving on a countable set of vertices V, with cardinality $|V|$. Their *positions* are denoted by $(x_1, \dots, x_n) \in V^n$. The set of functions $g : V^n \to \mathbb{R}$ is denoted by $\mathscr{C}_n$.

A *configuration* of n particles is the n-tuple of their positions modulo permutations of labels. More precisely, for $\mathbf{x} = (x_1, \dots, x_n) \in V^n$ the associated configuration is denoted by

$$\varphi(\mathbf{x}) := \sum_{i=1}^{n} \delta_{x_i} \tag{1}$$

where δ_z is the configuration having only one particle located at $z \in V$, i.e. for $y \in V$,

$$(\delta_z)_y = \begin{cases} 1 & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

This induces a map $\varphi : V^n \to \Omega_n$, where Ω_n is the set of configurations of n particles, i.e.

$$\Omega_n = \Big\{ \eta \in \chi^V : \|\eta\| = n \Big\}, \qquad \text{with} \qquad \|\eta\| := \sum_{x \in V} \eta_x$$

and define $\Omega = \{\eta \in \chi^V : \|\eta\| < \infty\}$ the set of finite particle configurations, namely $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$.

We denote by $\mathscr{E}_n$ the set of functions $f : \Omega_n \to \mathbb{R}$. In the above χ is the single-site state space. We shall consider both examples with a finite state space, such as the partial exclusion processes (with a restriction on the number of particles per site, i.e. $\chi = \{0, \dots, \alpha\}$ where $\alpha \in \mathbb{N}$ denotes the maximal number of particles per site), as well as examples with $\chi = \mathbb{N}$, such as the inclusion process or the independent random walk process.

With these preliminaries we next specify the distinction between a coordinate process and a configuration process.

Definition 2.1 (Coordinate process)**.** *We shall call a* coordinate process with n particles*, denoted by* $\{X^{(n)}(t) : t \geq 0\}$*, a stochastic process taking values in* V^n*. This describes the positions of particles in the course of time,* *i.e. for* $i = 1, \ldots, n$ *the random variable* $X_i^{(n)}(t)$ *denotes the position of the* i^{th} *particle at time* $t \geq 0$. *We denote by* $\{X(t), t \geq 0\}$ *a family of coordinate-processes* $\big(\{X^{(n)}(t) : t \geq 0\}, n \in \mathbb{N}\big)$, *labeled by the number of particles* $n \in \mathbb{N}$.

Throughout this paper we shall restrict to coordinate processes that are Markov processes.

Definition 2.2 (Configuration process). *We shall call a* configuration process, *denoted by* $\{\eta(t) : t \geq 0\}$, *a stochastic process taking values in* Ω. *This describes the sites occupancy numbers in the course of time, i.e. for* $i \in V$ *the random variable* $\eta_i(t)$ *denotes the number of particles at site* i *at time* $t \geq 0$.

Throughout this paper we shall restrict to configuration processes that conserve the number of particles, i.e. if the process $\{\eta(t) : t \geq 0\}$ is started from $\eta \in \Omega_n$ then $\eta(t) \in \Omega_n$ for all later times $t > 0$.

A configuration process is naturally induced by a coordinate process using the map φ defined in (1). There could be several coordinate processes whose image under the map φ yields the same configuration process. This leads us to the following definition.

Definition 2.3 (Compatibility). *A family of coordinate processes* $\{X(t) : t \geq 0\}$ *and a configuration process* $\{\eta(t) : t \geq 0\}$ *are compatible if there exists* $n \in \mathbb{N}$ *such that*

$$\{\varphi(X^{(n)}(t)) : t \geq 0\} = \{\eta(t) : t \geq 0\}$$

where the equality is in distribution.

Of course it is not guaranteed that starting from a Markov coordinate process the mapping φ defined in (1) induces a compatible configuration process that is also Markov. To further discuss this point we need to introduce the notion of permutation invariance. We denote by Σ_n the set of permutations of n elements. Moreover we define the operator U_φ mapping functions $f : \Omega_n \to \mathbb{R}$ to $U_\varphi f \in \mathscr{C}_n$ via $U_\varphi f = f \circ \varphi$.

Definition 2.4 (Permutation invariance).

a) A family of coordinate Markov processes $\{X(t) : t \geq 0\}$ *is said to be* permutation invariant *if, for every* $n \in \mathbb{N}$ *and for every permutation* $\sigma \in \Sigma_n$, *the processes* $\{(X_1^{(n)}(t), \ldots, X_n^{(n)}(t)) : t \geq 0\}$ *and* $\{(X_{\sigma(1)}^{(n)}(t), \ldots, X_{\sigma(n)}^{(n)}(t)) : t \geq 0\}$ *are equal in distribution.*

b) A function $g \in \mathscr{C}_n$ is said to be permutation invariant *if*

$$g(x_1, \ldots, x_n) = g(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \qquad \textit{for all} \quad \sigma \in \Sigma_n.$$

Equivalently, a function $g \in \mathscr{C}_n$ is permutation invariant if there exists a function $f : \Omega_n \to \mathbb{R}$ such that $g = U_\varphi g$.

c) A probability measure μ_n on V^n is called permutation invariant if, for all $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, under μ_n, the random vectors $(X_1, \ldots, X_n)$ and $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ have the same distribution.

We denote by $S_n(t)$ the infinitesimal generator of the n-particle coordinate process $\{X^{(n)}(t) : t \geq 0\}$, i.e., for $g \in \mathscr{C}_n$

$$S_n(t)g(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[g(X^{(n)}(t))]$$

where $\mathbb{E}_{\mathbf{x}}$ denotes expectation when the process is started from $\mathbf{x} \in V^n$. Let L_n denote its generator.

The following lemma shows that to permutation invariant coordinate processes one can naturally associate a compatible configuration process enjoying the Markov property.

Lemma 2.1. *Let $\{X(t) : t \geq 0\}$ be a family of permutation invariant coordinate Markov processes with generators L_n, $n \in \mathbb{N}$. Then it is possible to define an operator $\mathscr{L}$ acting on functions $f : \Omega \to \mathbb{R}$ as*

$$\mathscr{L} f(\eta) := L_n(U_\varphi f)(\mathbf{x}) \qquad \textit{for all} \quad \eta \in \Omega_n \ \textit{and} \ \mathbf{x} \in V^n \ : \ \varphi(\mathbf{x}) = \eta \qquad (2)$$

or, equivalently,

$$L_n U_\varphi = U_\varphi \mathscr{L} \qquad \textit{on} \quad \mathscr{E}_n. \qquad (3)$$

Then $\mathscr{L}$ is the infinitesimal generator of a Markov process $\{\eta(t), t \geq 0\}$ that is a configuration process compatible with $\{X(t) : t \geq 0\}$.

Proof. For a permutation $\sigma \in \Sigma_n$ we define the operator

$$T_\sigma g(x_1, \ldots, x_n) = g(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \qquad (4)$$

From the permutation invariance of the coordinate Markov processes $\{X_n(t) : t \geq 0\}$, $n \in \mathbb{N}$ it follows that

$$[L_n, T_\sigma] = 0 \qquad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \sigma \in \Sigma_n. \qquad (5)$$

Let $f : \Omega \to \mathbb{R}$ then, by definition, $U_\varphi f$ is a permutation-invariant function. Hence, from (5) it follows

$$T_\sigma L_n U_\varphi f = L_n T_\sigma U_\varphi f = L_n U_\varphi f \qquad \text{for all} \quad f \in \mathscr{E}_n. \qquad (6)$$

This means that $L_n U_\varphi f$ is permutation-invariant, hence there exists a function $\tilde{f} : \Omega \to \mathbb{R}$ such that

$$L_n[U_\varphi f](\mathbf{x}) = \tilde{f}(\varphi(\mathbf{x})) = U_\varphi \tilde{f}(\mathbf{x})$$

namely $L_n U_\varphi f = U_\varphi \tilde{f}$. Then it is possible to define the operator $\mathscr{L}$ acting on functions $f : \Omega \to \mathbb{R}$ such that $\mathscr{L} f = \tilde{f}$, and then (3) is satisfied. From (6) we have that

$$T_\sigma S_n(t) U_\varphi f = S_n(t) U_\varphi f \qquad \text{for all} \quad f \in \mathscr{E}_n \tag{7}$$

and then also $S_n(t)U_\varphi f$ is a permutation-invariant function at all times. Now, if we denote by $\mathscr{S}(t)$ the semigroup associated to $\mathscr{L}$, it follows that

$$\mathscr{S}(t) f(\eta) = S_n(t) U_\varphi f(\mathbf{x}) \qquad \text{for all} \quad \eta \in \Omega_n \ \text{ and } \ \mathbf{x} \in V^n \ : \ \varphi(\mathbf{x}) = \eta \tag{8}$$

namely

$$U_\varphi \mathscr{S}(t) = S_n(t) U_\varphi \qquad \text{on} \quad \mathscr{E}_n \tag{9}$$

From this it follows that $\mathscr{L}$ is the generator of a Markov process that is a configuration process compatible with $\{X(t) : t \geq 0\}$. $\square$

2.2 The annihilation operator

We continue by introducing the operators that remove particles either in the coordinate process (see (10) below) or in the configuration process (see (12), the so-called annihilation operator).

Definition 2.5 (Particle removal operators)**.** *We denote, for $n \in \mathbb{N}$, $1 \leq i \leq n$ the removal operator of the i^{th} labeled particle by $\pi_i^{(n)} : \mathscr{C}_{n-1} \to \mathscr{C}_n$, i.e., for $g \in \mathscr{C}_{n-1}$*

$$(\pi_i^{(n)} g)(x_1, \ldots, x_n) = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \qquad \textit{for all} \quad x_i \in V \tag{10}$$

We denote by $\Pi^{(n)} : \mathscr{C}_{n-1} \to \mathscr{C}_n$ the operator acting on $g \in \mathscr{C}_{n-1}$ via

$$\Pi^{(n)} g = \sum_{i=1}^{n} \pi_i^{(n)} g. \tag{11}$$

We also denote the total "annihilation operator" on configurations working on functions $f : \Omega \to \mathbb{R}$

$$\mathscr{A} f(\eta) = \sum_{x \in V} \eta_x f(\eta - \delta_x) \tag{12}$$

We remind the reader here that we are restricting to configurations $\eta \in \Omega$, i.e. with a finite number of particles and therefore the sum in (12) is a finite sum.

The annihilation operator is crucially important in the explanation of self-dualities for several particle systems [4, 5]. In particular, as it will be shown later, the fact that the generator $\mathscr{L}$ of the configuration process and the annihilation operator $\mathscr{A}$ commute, i.e. $[\mathscr{L}, \mathscr{A}] = 0$, is enough to obtain a self-duality when the process has a reversible measure. Such a commutation relation for the configuration process is equivalent to an intertwining relations between the coordinate process with n particles and the coordinate process with $n-1$ particles. This equivalence is the object of the next Theorem.

Theorem 2.1. *Let $\{X(t) : t \geq 0\}$ be a family of coordinate Markov processes with generators L_n, $n \in \mathbb{N}$ and a corresponding Markov configuration process $\{\eta(t) : t \geq 0\}$ with generator $\mathscr{L}$. Then the following statements are equivalent.*

a) The generators of the coordinate process with n and $n-1$ particles restricted to permutation invariant functions are intertwined via $\Pi^{(n)}$, i.e., for every $n \in \mathbb{N}$, and for all $g \in \mathscr{C}_{n-1}$ permutation invariant

$$[L_n \Pi^{(n)}](g) = [\Pi^{(n)} L_{n-1}](g) \tag{13}$$

b) The generator of the configuration process commutes with the total annihilation operator, i.e.,

$$[\mathscr{L}, \mathscr{A}] = 0 \tag{14}$$

Remark 2.1. *The probabilistic interpretation of* (13) *is as follows: if we remove a randomly chosen particle, then evolve the process, evaluate a permutation invariant function at time $t > 0$ and finally take expectation, then we can as well first evolve the process, remove a randomly chosen particle at time $t > 0$, evaluate the same permutation invariant function and take expectation. In other words, the operations "removing a randomly chosen particle" and "time evolution in the process followed by expectation" commute as long as we restrict to permutation invariant functions.*

Proof. We first show that

$$\Pi^{(n)} U_\varphi = U_\varphi \mathscr{A} \qquad \text{on} \quad \mathscr{E}_n. \tag{15}$$

This means that

$$[\Pi^{(n)}(f \circ \varphi)](\mathbf{x}) = \mathscr{A} f(\eta) \qquad \text{for all} \quad \eta \in \Omega_n \text{ and } \mathbf{x} \in V^n \; : \; \varphi(\mathbf{x}) = \eta \tag{16}$$

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\eta := \varphi(\mathbf{x}) = \sum_{i=1}^n \delta_{x_i}$, then we have

$$\varphi(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n) = \left(\sum_{i=1}^n \delta_{x_i} \right) - \delta_{x_l} = \eta - \delta_{x_l}$$

As a consequence:

$$\begin{aligned} [\Pi^{(n)}(f \circ \varphi)](\mathbf{x}) &= \sum_{l=1}^n f(\eta - \delta_{x_l}) \\ &= \sum_{x \in V} \eta_x f(\eta - \delta_x) \end{aligned} \tag{17}$$

where the last step follows because every $x \in V$ is counted exactly η_x times in the sum $\sum_{l=1}^n f(\eta - \delta_{x_l})$. This proves (15). Suppose now that $[\mathscr{L}, \mathscr{A}] = 0$, then, on $\mathscr{E}_n$ we have

$$L_{n-1}\Pi^{(n)}U_\varphi = L_{n-1}U_\varphi\mathscr{A} = U_\varphi\mathscr{L}\mathscr{A} = U_\varphi\mathscr{A}\mathscr{L} = \Pi^{(n)}U_\varphi\mathscr{L} = \Pi^{(n)}L_n U_\varphi$$

where the identities follow from (15), (3) and the commutation relation. Then (13) follows since, for all $g \in \mathscr{C}_n$ permutation invariant, $g = U_\varphi f$ for some $f \in \mathscr{E}_n$. The reverse implication is proved similarly. $\square$

2.3 Consistency

Definition 2.6 (Consistency).

a) *A family of coordinate Markov processes* $\{X(t) : t \geq 0\}$ *is said to be* consistent *if for every* $n \in \mathbb{N}$ *the processes* $\{(X_1^{(n)}(t), \dots, X_{n-1}^{(n)}(t)) : t \geq 0\}$ *and* $\{(X_1^{(n-1)}(t), \dots, X_{n-1}^{(n-1)}(t)) : t \geq 0\}$ *are equal in distribution.*

b) *A family of probability measures* μ_n *on* V^n*, indexed by* $n \in \mathbb{N}$*, is called* consistent *if for all* $n \geq 2$*, under* μ_n*, the distribution of* $(x_1, \dots, x_{n-1})$ *equals* μ_{n-1}.

Let $\{\mu_n, n \in \mathbb{N}\}$ be a consistent collection of probability measures on $V^n, n \in \mathbb{N}$. If additionally $\{\mu_n, n \in \mathbb{N}\}$ is permutation invariant, then *every* $n-1$ *dimensional marginal* of μ_n coincides with μ_{n-1}. A simple example of such a consistent permutation invariant family is

$$\mu_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \delta_{(x_{\sigma(1)}, \dots, x_{\sigma(n)})} \qquad \text{for some } \mathbf{x} \in V^n$$

Theorem 2.2. *Let $\{X(t) : t \geq 0\}$ be a family of coordinate Markov processes with generators L_n, $n \in \mathbb{N}$, and let $\{\eta(t) : t \geq 0\}$ be a compatible configuration process with generator $\mathscr{L}$. Assume that:*

i) $\mathscr{L}$ commutes with the annihilation operator $\mathscr{A}$.

ii) the probability measures $\{\mu_n, n \in \mathbb{N}\}$ on V^n, $n \in \mathbb{N}$, form a consistent family which is also permutation invariant.

Then we have consistency of the marginal coordinate processes starting from $\{\mu_n, n \in \mathbb{N}\}$, i.e., for all $n \in \mathbb{N}$, $g \in \mathscr{C}_{n-1}$ permutation invariant,

$$\mathbb{E}^{(n)}_{\mu_n}\left[g(X^{(n)}_1(t), \ldots, X^{(n)}_{n-1}(t))\right] = \mathbb{E}^{(n-1)}_{\mu_{n-1}}\left[g(X^{(n)}_1(t), \ldots, X^{(n)}_{n-1}(t))\right] \tag{18}$$

where $\mathbb{E}^n_{\mu_n}$ denotes expectation w.r.t. the Markov process $\{X^{(n)}(t) : t \geq 0\}$, started initially with distribution μ_n.

Proof. Let $g \in \mathscr{C}_{n-1}$ be a permutation invariant function, then we have $\pi^{(n)}_l g = \pi^{(n)}_k g$ for $k, l \in \{1, \ldots, n\}$. Then, by consistency and permutation invariance of μ_n, we have

$$\int \pi_l g(x_1, \ldots, x_n)\mu_n(dx_1 \ldots dx_n) = \int g(x_1, \ldots, x_{n-1})\mu_{n-1}(dx_1 \ldots dx_{n-1})$$

for all $n \in \mathbb{N}$ and all $l \in \{1, \ldots, n\}$. Therefore, using (13), we have

$$\begin{aligned}
\mathbb{E}^{(n)}_{\mu_n}\left[g(X^{(n)}_1(t), \ldots, X^{(n)}_{n-1}(t))\right] &= \mathbb{E}^{(n)}_{\mu_n}\left[\pi^{(n)}_n(g(X^{(n)}_1(t), \ldots, X^{(n)}_{n-1}(t), X^{(n)}_n(t))\right] \\
&= \int S_n(t)(\pi^{(n)}_n g)(x_1, \ldots, x_n)\mu_n(dx_1 \ldots dx_n) \\
&= \frac{1}{n}\int \left(S_n(t)\left(\sum_{k=1}^n \pi^{(n)}_k g\right)\right)(x_1, \ldots, x_n)\mu_n(dx_1 \ldots dx_n) \\
&= \frac{1}{n}\int S_n(t)(\Pi^{(n)} g)(x_1, \ldots, x_n)\mu_n(dx_1 \ldots dx_n) \\
&= \frac{1}{n}\int (\Pi^{(n)} S_{n-1}(t) g)(x_1, \ldots, x_n)\mu_n(dx_1 \ldots dx_n) \\
&= \int (S_{n-1}(t) g)(x_1, \ldots, x_{n-1})\mu_{n-1}(dx_1 \ldots dx_n) \\
&= \mathbb{E}^{(n-1)}_{\mu_{n-1}}\left[g(X^{(n)}_1(t), \ldots, X^{(n)}_{n-1}(t))\right]
\end{aligned} \tag{19}$$

this concludes the proof. □

3 Consistency and self-duality

In this section we show that consistency implies a form of self-duality whenever a process admits a strictly-positive reversible measure. We start by recalling the definition of *duality*.

Definition 3.1. *Let $\{Y_t\}_{t\geq 0}$, $\{\widehat{Y}_t\}_{t\geq 0}$ be two Markov processes with state spaces Ω and $\widehat{\Omega}$ and $D : \Omega \times \widehat{\Omega} \to \mathbb{R}$ a bounded measurable function. The processes $\{Y_t\}_{t\geq 0}$, $\{\widehat{Y}_t\}_{t\geq 0}$ are said to be dual with respect to D if*

$$\mathbb{E}_x\big[D(Y_t, \widehat{y})\big] = \widehat{\mathbb{E}}_{\widehat{y}}\big[D(y, \widehat{Y}_t)\big] \tag{20}$$

for all $y \in \Omega, \widehat{y} \in \hat{\Omega}$ and $t > 0$. In (20) $\mathbb{E}_y$ is the expectation with respect to the law of the $\{Y_t\}_{t\geq 0}$ process started at y, while $\widehat{\mathbb{E}}_{\widehat{y}}$ denotes expectation with respect to the law of the $\{\widehat{Y}_t\}_{t\geq 0}$ process initialized at $\widehat{y}$. We say that a process is self-dual when the dual process coincides with the original process.

It is useful to express the duality property in terms of generators of the two processes. If L denotes the generator of $\{Y_t\}_{t\geq 0}$ and $\widehat{L}$ denotes the generator of $\{\widehat{Y}_t\}_{t\geq 0}$, then (assuming that the duality functions are in the domain of the generators), the above definition is equivalent to $LD(\cdot, \widehat{y})(y) = \widehat{L}D(y, \cdot)(\widehat{y})$ where L acts on the first variable and $\widehat{L}$ acts on the second variable.
In order to prove our theorem on duality we recall two general results on self-duality from [5].

a) *Trivial duality function from a reversible measure.*

   If a Markov process $\{Y_t : t \geq 0\}$ with state-space Ω has a strictly-positive reversible measure ν, then the function $D : \Omega \times \Omega \to \mathbb{R}$ given by

$$D(x, y) = \frac{\delta_{x,y}}{\mu(x)} \tag{21}$$

   is a self-duality function.

b) *New duality functions via symmetries.*

   If D is a self-duality function and S is a symmetry of $\mathscr{L}$, then SD is a self-duality function.

Lemma 3.1. *Let $\mathscr{A}$ denote the annihilation operator defined in* (12)*. For $\xi, \eta \in \Omega$ denote by $F_\xi(\eta) = \delta_{\xi,\eta}$ the Kronecker delta. Then we have*

$$(e^{\mathscr{A}} F_\xi)(\eta) = \begin{cases} \prod_{x\in V} \frac{\eta_x!}{(\eta_x - \xi_x)!\xi_x!} & \textit{if } \xi \leq \eta \\ 0 & \textit{otherwise} \end{cases} \tag{22}$$

Proof. Let us define the "single-site annihilation operator" a as follows: for a function $f : \mathbb{N}_0 \to \mathbb{R}$ we define $af(n) := nf(n-1)$. Then the annihilation operator $\mathscr{A} = \sum_{x\in V} a_x$ where a_x denotes the operator a working on the variable η_x. Therefore, to prove (22), it is sufficient to prove that for all $n, k \in \mathbb{N}_0$

$$e^a F_k(n) = \begin{cases} \frac{n!}{(n-k)!k!} & \text{if } k \le n \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

with $F_k(n) = \delta_{k,n}$. For a function $f : \mathbb{N}_0 \to \mathbb{R}$ we have

$$e^a f(n) = \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} f(n-r).$$

Inserting $f = F_k$ gives (23). $\square$

Let ν be a strictly-positive function on Ω. We actually think of ν as a reversible measure, not necessarily a probability measure, but in this setting of countable state space Ω this simply coincides with a strictly positive function. We say that ν satisfies detailed balance w.r.t. the generator $\mathscr{L}$ of the configuration process if

$$\nu(\xi)\mathscr{L}(\xi,\eta) = \nu(\eta)\mathscr{L}(\eta,\xi). \tag{24}$$

If ν satisfies detailed balance, then the function

$$D_{\text{cheap}}(\xi,\eta) = \frac{\delta_{\xi,\eta}}{\nu(\xi)} \tag{25}$$

is a self-duality function, i.e.,

$$\mathscr{L} D_{\text{cheap}}(\xi,\cdot)(\eta) = \mathscr{L} D_{\text{cheap}}(\cdot,\eta)(\xi)$$

We then obtain the following result.

Theorem 3.1. *Let $\{X(t) : t \ge 0\}$ be a family of coordinate Markov processes with generators L_n, $n \in \mathbb{N}$, and let $\{\eta(t) : t \ge 0\}$ be a compatible configuration process with generator $\mathscr{L}$.*

a) Assume that one of the two equivalent conditions of Theorem 2.1 are satisfied. Assume furthermore that ν is a strictly-positive function on Ω, satisfying detailed balance w.r.t. $\mathscr{L}$. Then the function

$$D(\xi,\eta) = \begin{cases} \frac{1}{\nu(\xi)} \prod_{x\in V} \frac{\eta_x!}{(\eta_x-\xi_x)!\xi_x!} & \text{if } \xi \le \eta \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

is a self-duality function for the configuration process with generator $\mathscr{L}$.

b) Conversely, if ν is a strictly-positive function on Ω, satisfying detailed balance w.r.t. $\mathscr{L}$ and if (26) is a self-duality function, then item b) of Theorem 2.1 is satisfied, and as a consequence also item a) for a compatible coordinate process.

Proof. Item a) follows by acting with $e^{\mathscr{A}}$ on the η-variable in the cheap self-duality function (25) which produces (26) via Lemma 3.1. This produces a new self-duality function because, by assumption, $\mathscr{L}$ commutes with $\mathscr{A}$.

To prove item b) notice that the process with generator $\mathscr{L}$ conserves the total number of particles, and hence can be identified with a family of generators $\mathscr{L}_n$ working on functions $f : \Omega_n \to \mathbb{R}$ Now consider the case where $\eta \in \Omega_n$ and $\xi = \eta - \delta_x$, and $\xi \in \Omega_{n-1}$ for some $x \in V$. Then,

$$\nu(\xi) \cdot D(\xi, \eta) = \eta_x. \tag{27}$$

Because D is a self-duality function and ν is a strictly-positive function on Ω, satisfying detailed balance w.r.t. $\mathscr{L}$, the operator $K : C(\Omega_{n-1}, \mathbb{R}) \to C(\Omega_n, \mathbb{R})$

$$Kf(\eta) := \sum_{\xi \in \Omega_{n-1}} \nu(\xi) D(\xi, \eta) f(\xi)$$

intertwines between $\mathscr{L}_n$ and $\mathscr{L}_{n-1}$, i.e., $K(\mathscr{L}_{n-1} f) = \mathscr{L}_n K f$, see [7] for the connection between duality functions and corresponding intertwining kernel operators. Now, via (27) we obtain

$$Kf(\eta) = \sum_x \eta_x f(\eta - \delta_x) = \mathscr{A} f(\eta)$$

and we conclude $\mathscr{A}(\mathscr{L}_{n-1} f) = \mathscr{L}_n \mathscr{A} f$, which is exactly the commutation property $[\mathscr{L}, \mathscr{A}] = 0$. □

4 Consistency for systems with absorbing sites

In this section we study consistency for systems of interacting particles with absorbing sites. These systems are important as they emerge as dual of particles systems with reservoirs that are prototypical models for mass-transport out-of-equilibrium. In Section 5 we will analyse the consequences of consistency in the context of non-equilibrium systems.

Let $\{\eta(t) : t \geq 0\}$ denote a configuration process with generator $\mathscr{L}$. Let $V^* \supset V$ be a countable set of sites containing V. We will define a configuration process on $\mathbb{N}^{V^*}$ as follows: inside V the particles move according to the generator $\mathscr{L}$, and additionally every particle at site i moves with rate $r(i,j)$ to a site $j \in V^* \setminus V$, independently from each other. Particles arriving at sites $j \in V^* \setminus V$ are absorbed, i.e., do not move anymore. We thus obtain what we call the process with absorbing sites $V^* \setminus V$ and absorption rates $r(i,j)$. The generator of this process is then given by

$$\mathscr{L}^{abs} f(\eta) = \mathscr{L} f(\eta) + H f(\eta) \qquad \text{for} \quad f : \mathbb{N}^{V^*} \to \mathbb{R} \tag{28}$$

where $\mathscr{L}$ only works on the variables $\{\eta_x, x \in V\}$ and where H denotes the absorption part of the generator, i.e.,

$$H f(\eta) = \sum_{i \in V, j \in V^* \setminus V} r(i,j) \eta_i \left(f(\eta - \delta_i + \delta_j) - f(\eta) \right) \tag{29}$$

We can rewrite H as follows

$$H = \sum_{i \in V, j \in V^* \setminus V} r(i,j)(a_i a_j^\dagger - a_i a_i^\dagger) \tag{30}$$

where the creation and annihilation operators $a_i^\dagger, a_i$ are defined via

$$\begin{aligned} a_i^\dagger f(\eta) &= f(\eta + \delta_i) \\ a_i f(\eta) &= \eta_i f(\eta - \delta_i) \end{aligned} \tag{31}$$

A process with generator (28) is called an absorbing extension of the generator $\mathscr{L}$.

Definition 4.1.

- *Let $\{\eta(t) : t \geq 0\}$ be a configuration process on the lattice V with generator $\mathscr{L}$ then we define its absorbing extension $\{\eta^{abs}(t) : t \geq 0\}$ to the lattice V^* the coordinate process with generator $\mathscr{L}^{abs}$ given by* (28).
- *Let $\{X(t) : t \geq 0\}$ be a family of coordinate Markov processes on the lattice V. Then we define its absorbing extension $\{X^{abs}(t) : t \geq 0\}$ to the lattice V^* the family of coordinate Markov processes $\{X^{abs,(n)}(t) : t \geq 0\}$, $n \in \mathbb{N}$ on $(V^*)^n$ defined by adding to the jumps of $\{X^{(n)}(t) : t \geq 0\}$, $n \in \mathbb{N}$ the additional jumps, independently for each particle from i to j at rate $r(i,j)$, $i \in V, j \in V^* \setminus V$.*

Lemma 4.1. *Let $\{X(t) : t \geq 0\}$ be a family of coordinate Markov processes on the lattice V and $\{\eta(t) : t \geq 0\}$ be a compatible configuration process. Let $\{X^{abs}(t) : t \geq 0\}$ and $\{\eta^{abs}(t) : t \geq 0\}$ be the respective absorbing extensions on the lattice V^*, then $\{\eta^{abs}(t) : t \geq 0\}$ in compatible with $\{X^{abs}(t) : t \geq 0\}$.*

Proof. It follows immediately from the definition of absorbing extensions. □

We then have the following results.

Lemma 4.2. *Let $\mathscr{A}$ denote the annihilation operator* (12)*. If* $[\mathscr{L}, \mathscr{A}] = 0$*, then for every absorbing extension* (28)*, we have that*

$$[\mathscr{L}^{abs}, \mathscr{A}^{abs}] = 0 \qquad \textit{for} \quad \mathscr{A}^{abs} f(\eta) := \sum_{i \in V^*} a_i \tag{32}$$

with a_i as defined in (31)*.*

Proof. Since $[\mathscr{L}, \mathscr{A}] = 0$ by assumption, we only have to prove that

$$[H, \mathscr{A}^{abs}] = 0$$

Using (30) and the fact that operators working on variables at different sites commute, we have to show that for all $i, j \in V^*$

$$[a_i a_j^\dagger - a_i a_i^\dagger, a_i + a_j] = 0$$

This in turn follows from the commutation relations $[a_i, a_j] = 0, [a_i, a_j^\dagger] = \delta_{i,j}$. □

Theorem 4.1. *Let $\{X(t) : t \geq 0\}$ be a family of coordinate Markov processes and let $\{\eta(t) : t \geq 0\}$ be a compatible configuration process with generator $\mathscr{L}$. Assume that $[\mathscr{L}, \mathscr{A}] = 0$. Then the absorbing extension $\{X^{abs}(t) : t \geq 0\}$ of $\{X(t) : t \geq 0\}$ on V^* is consistent if started from a consistent family $\{\mu_n, n \in \mathbb{N}\}$ of probability measures on $(V^*)^n$, $n \in \mathbb{N}$ which is also permutation invariant. Namely for all $n \in \mathbb{N}$, $f : (V^*)^n \to \mathbb{R}$ permutation invariant,*

$$\mathbb{E}^{(n)}_{\mu_n}\left[f(X_1^{abs,(n)}(t), \ldots, X_{n-1}^{abs,(n)}(t))\right] = \mathbb{E}^{(n-1)}_{\mu_{n-1}}\left[f(X_1^{abs,(n)}(t), \ldots, X_{n-1}^{abs,(n)}(t))\right] \tag{33}$$

where $\mathbb{E}^n_{\mu_n}$ denotes expectation w.r.t. the Markov process $\{X^{abs,(n)}(t) : t \geq 0\}$, started initially with distribution μ_n.

Proof. It follows from Lemma 4.1, Lemma 4.2 and Theorem 2.2. □

Corollary 4.1. *Assume, in the setting of Theorem 4.1 that both V and V^* are finite sets. Assume that $\{\eta(t) : t \geq 0\}$ is a configuration process on the lattice V and $\{\eta^{abs}(t) : t \geq 0\}$ an absorbing extension to the lattice V^*. Let $\zeta : V^* \setminus V \to \mathbb{N}_0$ be a configuration on the set of absorbing sites. Denote by*

$$q(\eta, \zeta) := \lim_{t\to\infty} \mathbb{P}^{abs}_{\eta} \left(\eta(t) = \zeta\right) \tag{34}$$

the corresponding absorption probabilities, i.e., the probability that eventually $\eta(t)$ settles in the absorbing configuration ζ. Then we have the following family of recursion relations for the absorption probabilities:

$$\sum_{i\in V^*\setminus V} \eta_i q(\eta - \delta_i, \zeta) = \sum_{i\in V^*\setminus V} (\zeta_i + 1) q(\eta, \zeta + \delta_i) \tag{35}$$

Proof. Using (32) we have that also the semigroup of $\{\eta^{abs}(t), t \geq 0\}$ commutes with $\mathscr{A}$, then we obtain

$$\sum_{i\in V^*} \eta_i \mathbb{E}_{\eta-\delta_i}[f(\eta(t)] = \sum_{i\in V^*} \mathbb{E}_{\eta}[\eta_i(t) f(\eta(t) - \delta_i)] \tag{36}$$

Applying this for the function $f(\eta) = \mathbf{1}_{\{\eta=\zeta\}}$ and taking the limit $t \to \infty$ gives (35). □

Remark 4.1. *The recursion relations* (35) *express combinations of absorption probabilities from the initial configuration η in terms of combinations of absorption probabilities from an initial configuration η' with one particle less. Usually, the absorption probabilities with only one particle initially can be obtained in closed form. As we will see below, although these equations are not sufficient to determine in closed form the absorption probabilities from the single particle absorption probabilities, they are still putting severe restrictions on the absorption probabilities.*

4.1 Recursion relations for absorption probabilities on chains

An important particular case is the setting in which we have a chain $V^* = \{0, 1, \ldots, N, N+1\}$ and where the vertices $V^* \setminus V = \{0, N+1\}$ are the absorbing sites. Such chains arise as duals of non-equilibrium systems driven by reservoirs at left and right ends. If in this setting, the absorption probabilities can be be computed, then we have full control of the moments for dual systems out-of-equilibrium (see next section for more details and examples).

In this section we show how, in systems with absorbing sites, the consistency property imposes significant restrictions on the absorption probabilities, and, as a consequence, on the non-equilibrium steady state duality-moments. Consider a consistent configuration process $\{\eta(t) : t \geq 0\}$ where particles move on the vertex set $\{0, 1, \dots, N+1\}$ with $0, N+1$ absorbing states. For $\eta \in \Omega$, $|\eta| = m$, for $\kappa = 0, \dots, m$, $i \in \{0, \dots, N+1\}$, we define the probabilities

$$q_\eta^{(i)}(\kappa, t) := \mathbb{P}_\eta\left(\eta_i(t) = \kappa\right)$$

and the absorption probabilities

$$q_\eta^{(i)}(\kappa) := q(\eta, \kappa\delta_0 + (m-\kappa)\delta_{\kappa+1}) = \lim_{t\to\infty} q_\eta^{(i)}(\kappa, t)$$

Notice that, because $i \in \{1, \dots, N\}$ are not absorbing, $q_\eta^{(i)}(\kappa) = 0$, $i \in \{1, \dots, N\}$, and by conservation of the total number of particles, $q_\eta^{(N+1)}(\kappa) = q_\eta^{(0)}(m-\kappa)$.

The following theorem then gives a general recursion relation for the absorption probabilities.

Theorem 4.2. *Let $i \in \{0, 1, \dots, N+1\}$, $\eta \in \Omega$, and fix $m := |\eta|$. For $\kappa = 0, \dots, m-1$ we define*

$$A_\eta^{(i)}(\kappa, t) := \frac{1}{m-\kappa} \sum_{j=0}^{N+1} \eta_j \cdot \mathbb{E}_{\eta-\delta_j}\left[\binom{\eta_i(t)}{\kappa} \cdot \mathbf{1}_{\eta_i(t)\geq\kappa}\right]$$

then

$$\sum_{n=\kappa}^{m} \binom{n}{\kappa} \cdot q_\eta^{(i)}(n, t) = A_\eta^{(i)}(\kappa, t), \qquad \kappa = 0, \dots, m-1 \tag{37}$$

These are m independent equations for the $m+1$ unknown $\{q_\eta^{(i)}(n, t),\ n = 0, \dots, m\}$. Notice that for $\kappa = 0$ the condition above gives $\sum_{n=0}^m q_\eta^{(i)}(n, t) = 1$.

Proof. Fix $\kappa \in \{0, 1, \dots, m-1\}$, we use (36) with

$$f_\kappa^{(i)}(\eta) := \binom{\eta_i}{\kappa} \cdot \mathbf{1}_{\eta_i\geq\kappa} \tag{38}$$

then, applying consistency of the absorbing configuration process, i.e.,

$$\sum_{j=0}^{N+1} \mathbb{E}_\eta(\eta_j(t) f(\eta(t) - \delta_j) = \sum_{i=0}^{N+1} \eta_j \mathbb{E}_{\eta-\delta_j}(f(\eta(t)))$$

for $f = f_\kappa^{(i)}$ gives

$$\sum_{j\neq i} \mathbb{E}_\eta \left[\eta_j(t) \binom{\eta_i(t)}{\kappa} \cdot \mathbf{1}_{\eta_i(t)\geq\kappa} \right] + \mathbb{E}_\eta \left[\eta_i(t) \binom{\eta_i(t)-1}{\kappa} \cdot \mathbf{1}_{\eta_i(t)\geq\kappa+1} \right]$$

$$= \sum_{j=0}^{N+1} \eta_j \mathbb{E}_{\eta-\delta_j} \left[\binom{\eta_i(t)}{\kappa} \cdot \mathbf{1}_{\eta_i(t)\geq\kappa} \right] = (m-\kappa) A_\eta^{(i)}(\kappa, t) \tag{39}$$

The left-hand side in (39) is equal to

$$\mathbb{E}_\eta \left[(m-\eta_i(t)) \binom{\eta_i(t)}{\kappa} \cdot \mathbf{1}_{\eta_i(t)\geq\kappa} \right] + \mathbb{E}_\eta \left[\eta_i(t) \binom{\eta_i(t)-1}{\kappa} \cdot \mathbf{1}_{\eta_i(t)\geq\kappa+1} \right]$$

$$= \sum_{n=\kappa}^{m} (m-n) \binom{n}{\kappa} \cdot \mathbb{P}_\eta \left(\eta_i(t) = n \right) + \sum_{n=\kappa+1}^{m} n \binom{n-1}{\kappa} \cdot \mathbb{P}_\eta \left(\eta_i(t) = n \right)$$

$$= (m-\kappa) \mathbb{P}_\eta \left(\eta_i(t) = \kappa \right) + \sum_{n=\kappa+1}^{m} \left((m-n) \binom{n}{\kappa} + n \binom{n-1}{\kappa} \right) \cdot \mathbb{P}_\eta \left(\eta_i(t) = n \right)$$

$$= (m-\kappa) \mathbb{P}_\eta \left(\eta_i(t) = \kappa \right) + (m-\kappa) \sum_{n=\kappa+1}^{m} \binom{n}{\kappa} \cdot \mathbb{P}_\eta \left(\eta_i(t) = n \right)$$

$$= (m-\kappa) \sum_{n=\kappa}^{m} \binom{n}{\kappa} \cdot q_\eta^{(i)}(n, t)$$

hence, from (39) we get (37).

□

Illustration: Independent random walkers

As an illustration, consider the case of a system of independent random walkers starting from a configuration of the type $\eta = m\delta_u$, then we define the random variable $X^{(m)}$:=number of particles eventually absorbed at site 0, then $X^{(m)} \sim$Bin(p, m) with $p = 1 - \frac{u}{N+1}$ then, in this case

$$\begin{aligned} A_\eta^{(i)}(\kappa) \quad &:= \quad \frac{1}{m-\kappa} \sum_{j=0}^{N+1} \eta_j \cdot \mathbb{E}_{\eta-\delta_j} \left[\binom{\eta_i(\infty)}{\kappa} \cdot \mathbf{1}_{\eta_i(\infty)\geq\kappa} \right] \\ &= \quad \frac{m}{m-\kappa} \, \mathbb{E}_{(m-1)\delta_u} \left[\binom{\eta_i(\infty)}{\kappa} \cdot \mathbf{1}_{\eta_i(\infty)\geq\kappa} \right] \end{aligned} \tag{40}$$

hence the relation (37) is equivalent to the relation

$$\mathbb{E} \left[\binom{X^{(m)}}{\kappa} \right] = \frac{m}{m-\kappa} \, \mathbb{E} \left[\binom{X^{(m-1)}}{\kappa} \right] \tag{41}$$

that is indeed true because, if $X^{(m)}$ ∼Bin(p, m) then

$$\mathbb{E}\left[\binom{X^{(m)}}{\kappa}\right] = p^\kappa \binom{m}{\kappa} \tag{42}$$

Proof of (42):

$$\begin{aligned}
\mathbb{E}\left[\binom{X^{(m)}}{\kappa}\right] &= \sum_{n=\kappa}^{m} \binom{n}{\kappa}\binom{m}{n} p^n (1-p)^{m-n} \\
&= \binom{m}{\kappa} \sum_{n=\kappa}^{m} \binom{m-\kappa}{m-n} p^n (1-p)^{m-n} = \\
&= \binom{m}{\kappa} p^\kappa \sum_{s=0}^{m-\kappa} \binom{m-\kappa}{s} p^{m-\kappa-s}(1-p)^s = \binom{m}{\kappa} p^\kappa
\end{aligned} \tag{43}$$

5 Correlation functions in non-equilibrium steady states

In [1] we have considered several non-equilibrium systems where the vertex set is the set $V = \{1, \dots, N\}$ and where the left and right boundaries are coupled to reservoirs. Let us denote $\{\eta(t) : t \geq 0\}$ a process of this type. Assume that the process has a unique stationary measure (called non-equilibrium steady state) denoted by μ_{LR}.

The dual system is a particle system with two extra vertices $V^* = \{0, 1, \dots, N+1\}$ where the sites $0, N+1$ are absorbing. In the systems considered in [1] the dual particle systems which appear are symmetric inclusion, symmetric exclusion or symmetric independent random walkers (or thermalizations of these processes such as the discrete KMP model). As a consequence, the generator of these dual processes commutes with the annihilation operator [4], and as showed in Theorem 4.1, the same holds for every absorbing extension.

The duality functions between the original process $\{\eta(t) : t \geq 0\}$ and the absorbing dual $\{\xi(t) : t \geq 0\}$ are of the form

$$D(\xi, \eta) = \rho_L^{\xi_0} \rho_R^{\xi_{N+1}} \prod_{i=1}^{N} d(\xi_i, \eta_i) \tag{44}$$

where ρ_L, ρ_R are the densities (or temperatures in case the staring system is of the type BEP, BMP) associated to the left and right reservoir, and where $d(k, \cdot)$ is the single-site duality function in the bulk. We then have the following formula relating the expectations of the duality functions in the non-equilibrium steady state of the process $\{\eta(t) : t \geq 0\}$ with the absorption probabilities in the dual process $\{\xi(t) : t \geq 0\}$.

$$\int D(\xi,\eta)\mu_{LR}(d\eta) = \sum_{k,l:k+l=|\xi|} \rho_L^k \rho_R^l \mathbb{P}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}) \tag{45}$$

Notice that in all the cases treated in [1], the function $D(\xi,\eta)$ is a polynomial in the variables $\eta_i, i \in V$ of degree $|\xi| = \sum_{i\in V} \xi_i$.

We can now combine the consistency relations (36) with (45) in order to obtain general properties of expectations of duality functions, under the only assumption that the dual system has the consistency property. We further assume that the dual system has absorption rates 1 for each particle. This fixes completely the rates associated to the reservoirs (see [1]) in the original $\{\eta(t) : t \geq 0\}$ system. Notice now that for the dual system, the dual process, and hence the absorption probabilities are no longer depending on the parameters ρ_L, ρ_R associated to the reservoirs, i.e., all information about the reservoirs is contained in the duality function.

An important special case is when for the dual system, the single particle motion is *symmetric nearest neighbor random walk on* $\{1, \dots, N\}$. This implies that the absorbed version, i.e., symmetric nearest neighbor random walk on $\{0, \dots, N+1\}$ with absorption at both ends has absorption probabilities in closed form given by

$$\mathbb{P}_i(X^{abs}(\infty) = N+1) = 1 - \mathbb{P}_i(X^{abs}(\infty) = 0) = \frac{i}{N+1} \tag{46}$$

5.1 The two-point correlation function

We start with the following general result on the two-point correlation function in the non-equilibrium steady state μ_{LR}.

Theorem 5.1. *For $s, r \in \{1, \dots, N\}$ and $s \neq r$ we have*

$$\begin{aligned} &\mathbf{cov}(\eta_r,\eta_s) = \mathbb{E}_{\mu_{LR}}[\eta_s\eta_r] - \mathbb{E}_{\mu_{LR}}[\eta_r]\mathbb{E}_{\mu_{LR}}[\eta_s] \\ = \;& \frac{1}{2}(\rho_L-\rho_R)^2 \left(\mathbb{P}^{irw}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0+\delta_{N+1}) - \mathbb{P}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0+\delta_{N+1})\right) \end{aligned}$$

where $\mathbb{P}^{irw}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0 + \delta_{N+1})$ denotes the absorption probability for two independent particles *initially starting from the configuration $\delta_r + \delta_s$.*

PROOF. Applying the recursion (35) for the dual absorbing system gives, for $0 \le k + l \le |\xi| - 1$,

$$\begin{aligned}\sum_i \xi_i \mathbb{P}_{\xi-\delta_i}(\xi(\infty) = k\delta_0 + l\delta_{N+1}) &= (k+1)\mathbb{P}_\xi(\xi(\infty) = (k+1)\delta_0 + l\delta_{N+1}) \\ &+ (l+1)\mathbb{P}_\xi(\xi(\infty) = k\delta_0 + (l+1)\delta_{N+1})\end{aligned}$$

where we denoted $\xi(\infty) = \lim_{t\to\infty} \xi(t)$ the limiting random state in which the initial ξ fixates. Now let us consider the particular case $\xi = \delta_r + \delta_s$, and $k = 1, l = 0$. Then from (47) we obtain

$$\begin{aligned}&\mathbb{P}_{\delta_s}(\xi_\infty = \delta_0) + \mathbb{P}_{\delta_r}(\xi_\infty = \delta_0) \\ =\ & 2\mathbb{P}_{\delta_s+\delta_r}(\xi_\infty = 2\delta_0) + \mathbb{P}_{\delta_s+\delta_r}(\xi_\infty = \delta_0 + \delta_{N+1}.\end{aligned} \tag{47}$$

Choosing $\xi = \delta_r + \delta_s$, and $k = 0, l = 1$ gives

$$\begin{aligned}&\mathbb{P}_{\delta_s}(\xi_\infty = \delta_{N+1}) + \mathbb{P}_{\delta_r}(\xi_\infty = \delta_{N+1}) \\ =\ & 2\mathbb{P}_{\delta_s+\delta_r}(\xi_\infty = 2\delta_{N+1}) + \mathbb{P}_{\delta_s+\delta_r}(\xi_\infty = \delta_0 + \delta_{N+1})\end{aligned} \tag{48}$$

Note that equations (47) and (48) are not independent, indeed summing up rhs and lhs of both equations yields the trivial identity $2 = 2$. Let us abbreviate

$$\mathbb{P}_{\delta_u}(\xi_\infty = \delta_0) = 1 - p_u \tag{49}$$

Then we have

$$\mathbb{P}^{irw}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0 + \delta_{N+1}) = p_s + p_r - 2p_s p_r \tag{50}$$

Using (47)-(48) together with (49) we obtain

$$\begin{aligned}\mathbb{E}_{\mu_{LR}}[\eta_s\eta_r] &= \rho_L^2 \mathbb{P}_{\delta_s+\delta_r}(\xi(\infty) = 2\delta_0) + \rho_R^2 \mathbb{P}_{\delta_s+\delta_r}(\xi(\infty) = 2\delta_{N+1}) \\ &+ \rho_L\rho_R \mathbb{P}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0 + \delta_{N+1}) \\ &= \frac{1}{2}\rho_L^2((1-p_s) + (1-p_r)) + \frac{1}{2}\rho_R^2(p_s + p_r) \\ &- \frac{1}{2}\mathbb{P}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0 + \delta_{N+1})(\rho_L - \rho_R)^2\end{aligned}$$

Now using $\mathbb{E}_{\mu_{LR}}[\eta_u] = \rho_L + p_u(\rho_R - \rho_L)$ and (50) we obtain the claimed formula (47) for the covariance. □

REMARK 5.1.

a) If the $\{\eta(t) : t \geq 0\}$ is the system of independent symmetric random walkers with reservoirs, then also the dual absorbing walkers $\{\xi(t) : t \geq 0\}$ are independent symmetric random walkers and then of course we have from Theorem 5.1 zero covariance in the non-equilibrium steady state. This is consistent with the fact that in that case the non-equilibrium steady state μ_{LR} is a product of Poisson measures, with at location x expectation $\rho_x = \rho_L(1 - p_x) + \rho_R p_x$, where p_x is the single particle absorption probability from (49).

b) In the general case (i.e., only assuming consistency of the absorbing dual system) we obtain that the covariance is exactly *quadratic in $(\rho_L - \rho_R)$ and the multiplying factor equals a factor minus $\frac{1}{2}$ times the difference between $\mathbb{P}_{\delta_s+\delta_r}(\xi(\infty) = \delta_0 + \delta_{N+1})$ and the independent random walk expression of the same quantity (expressions which are both not depending on ρ_L, ρ_R). This quantity is non-positive for exclusion particles (by Liggett's inequality [8], chapter 8) and non-negative for inclusion particles by the analogue of Ligget's inequality from [6].*

5.2 Expectation of higher order moments

For higher moments, we exploit once more the consistency for the dual absorbing system. For a dual configuration ξ with support in $\{1, \ldots, N\}$ and with $|\xi| = m$, the duality polynomial $D(\xi, \eta)$ is a multivariate polynomial of degree m in the $\eta_i, i = 1, \ldots, N$ variables, and for its limiting expectation we obtain, using duality with the absorbing system (45):

$$M_{LR}(\xi) := \mathbb{E}_{\mu_{LR}}[D(\xi, \cdot)] = \rho_R^m \sum_{k=0}^{m} q_\xi^0(k) \left(\tfrac{\rho_L}{\rho_R}\right)^k \tag{51}$$

where we remind the notation $q_\xi^0(k)$ for the probability that starting from ξ in the dual system, eventually k particles are absorbed at 0. We now observe that this can be written as $M_{LR}(\xi) = \rho_R^m \mathbb{E}_\xi\left[(\rho_L/\rho_R)^{\xi_0(\infty)}\right]$ so we rewrite this as

$$M_{LR}(\xi) = \rho_R^m G\big(\xi, \tfrac{\rho_L}{\rho_R}\big) \qquad \text{with} \qquad G(\xi, z) := \mathbb{E}_\xi\left[z^{\xi_0(\infty)}\right] \tag{52}$$

$G(\xi, \cdot)$ being the generating function of the number of absorbed particles at 0 starting from the configuration ξ. Here we define as usual $G(\xi, 0)$ by continuous extension, i.e., $G(\xi, 0) := \lim_{z\to 0} G(\xi, z) = \mathbb{P}_\xi(\xi_0(\infty) = 0)$, which is equal to the probability that all the particles in ξ are eventually absorbed at $N + 1$. We then have the following recursion theorem.

Theorem 5.2. *For all dual configurations ξ with support in $\{1,\dots,N\}$ and with $|\xi| = m$, we have*

$$(1-z)G'(\xi,z) + mG(\xi,z) = \sum_{j=1}^{N} \xi_j G(\xi - \delta_j, z) \tag{53}$$

and then

$$G(\xi,z) = (1-z)^m G(\xi,0) + (1-z)^m \sum_{j=1}^{N} \xi_j \int_0^z \frac{1}{(1-u)^{m+1}} G(\xi-\delta_j,u)du \tag{54}$$

Proof. Because of commutation of the generator of the absorbing system with the annihilation operator (12) we can write

$$\lim_{t\to\infty} \mathbb{E}_\xi[(\mathscr{A} z^{\xi_0})(t)] = \sum_{j=1}^{N} \xi_j G(\xi - \delta_j, z)$$

which leads to

$$\mathbb{E}_\xi[\xi_0(\infty) z^{\xi_0(\infty)-1}] + \mathbb{E}_\xi\left[(m - \xi_0(\infty)) z^{\xi_0(\infty)}\right] = \sum_{j=1}^{N} \xi_j G(\xi - \delta_j, z)$$

which is (53). We can now "integrate" the recursion (53) as follows. Putting $G(\xi,z) = (1-z)^m H(\xi,z)$ and substituting in (53) we find,

$$(1-z)^{m+1} H'(\xi,z) = \sum_{j=1}^{N} \xi_j G(\xi - \delta_j, z)$$

and by noticing that $H(\xi,0) = G(\xi,0) = \mathbb{P}_\xi(\xi_0(\infty) = 0)$, by integrating we obtain (54). □

The recursion (54) can be iterated until we are left with one particle configurations, for which we can use, in the nearest-neighbor case (46)

$$G(\delta_x, z) = \frac{x}{N+1} + \left(1 - \frac{1}{N+1}\right) z$$

The recursion (53), or equivalently, its integrated form (54) implies that the knowledge of the probability that all particles in ξ are absorbed at $N+1$, i.e., $G(\xi,0)$ suffices to obtain the full generating function $G(\xi,z)$, and, as a consequence, the expectations of all duality functions $\int D(\xi,\eta)\mu_{LR}(d\eta)$ in the non-equilibrium steady state μ_{LR}.

6 Examples

In this section, we verify explicitly the recursion (53) for three exactly solvable cases.

6.1 Independent simple random walkers

In this case, the stationary steady state is a product of Poisson distributions, so we can explicitly verify the recursion (53). Indeed, in that case, we have, for a configuration ξ with m particles we have

$$G(\xi, z) = \prod_i \left(z\left(1-p_i\right)+p_i\right)^{\xi_i}$$

where p_i is the probability to be absorbed at the right end, when starting from i. As a consequence,

$$\begin{aligned}
& (1-z)G'(\xi, z) \\
= \;& (1-z)\sum_{j=1}^{N}\Bigg(\xi_j\left(1-p_j\right)\left(z\left(1-p_j\right)+p_j\right)^{\xi_j-1} \\
& \prod_{i=1, i\neq j}^{N}\left(z\left(1-p_i\right)+p_i\right)^{\xi_i}\Bigg) \\
= \;& \sum_{j=1}^{N}\Bigg(\xi_j\left(-z\left(1-p_j\right)-p_j+1\right)\left(z\left(1-p_j\right)+p_j\right)^{\xi_j-1} \\
& \prod_{i=1, i\neq j}^{N}\left(z\left(1-p_i\right)+p_i\right)^{\xi_i}\Bigg) \\
= \;& -mG(\xi, z)+\sum_{j=1}^{N}\xi_j G(\xi-\delta_j, z)
\end{aligned}$$

Remark 6.1. *As a further application of the recursion* (53) *in the same spirit, one can easily show by induction that if the probabilities for all the particles to be absorbed at zero factorize, i.e., if* $G(\xi, 0) = \prod_i G(\delta_i, 0)^{\xi_i}$ *for all* ξ, *then the generating function factorizes, i.e.,* $G(\xi, z) = \prod_i G(\delta_i, z)^{\xi_i}$ *and as a consequence the system has a product invariant non-equilibrium steady state* μ_{LR}.

6.2 Exclusion

For exclusion process the matrix product ansatz gives an algebraic procedure to calculate all correlation functions. This provides a recursion relation for the correlation functions (formula (A.7) in [2]) that reads

$$\begin{aligned}\mathbb{E}^N_{\mu_{LR}}[\eta_{i_1}\eta_{i_2}\dots\eta_{i_m}] &= (\rho_a-\rho_b)\Big(1-\frac{i_m}{N+1}\Big)\mathbb{E}^{N-1}_{\mu_{LR}}[\eta_{i_1}\eta_{i_2}\dots\eta_{i_{m-1}}] \\ &+ \rho_b\mathbb{E}^{N}_{\mu_{LR}}[\eta_{i_1}\eta_{i_2}\dots\eta_{i_{m-1}}] \end{aligned} \tag{55}$$

where $\mathbb{E}^N_{\mu_{LR}}$ denotes expectation in the non-equilibrium steady states of a system of size N. In this section ξ will denote a configuration of m dual particles placed at positions $1 \le i_1 < i_2 < \dots < i_m \le N$, i.e.

$$\xi = \sum_{j=1}^{m} \delta_{i_j} \tag{56}$$

Duality yields

$$\mathbb{E}^N_{\mu_{LR}}[\eta_{i_1}\eta_{i_2}\dots\eta_{i_m}] = \sum_{k=0}^{m} \rho_a^k \rho_b^{m-k} p^N_\xi(k) \tag{57}$$

where $p^{(N)}_\xi(k)$, with $k \in \{0,1,\dots,m\}$, denotes the probability that k dual particles are eventually absorbed at 0 when initially m dual particles start from $i_1,\dots,i_m$ in a system of size N.

Inserting (57) in (55) the principle of identity of polynomials turns the recurrence relation for the correlation functions into a recurrence relation for the absorption probabilities:

$$\begin{aligned} p^{(N)}_\xi(k) &= \mathbf{1}_{k\neq 0}\cdot\Big[p^{(N-1)}_{\xi-\delta_{i_m}}(k-1)p^{(N)}_{\delta_{i_m}}(1)\Big] \\ &+ \mathbf{1}_{k\neq m}\cdot\Big[p^{(N)}_{\xi-\delta_{i_m}}(k) - p^{(N-1)}_{\xi-\delta_{i_m}}(k)p^{(N)}_{\delta_{i_m}}(1)\Big] \end{aligned} \tag{58}$$

where $k \in \{0,1,2,\dots,m\}$. Introducing the generating function

$$G^{(N)}_\xi(z) = \sum_{k=0}^{m} p^{(N)}_\xi(k)\, z^k \tag{59}$$

the recursion relation of the absorption probabilities (58) implies the recursion relation for the generating function

$$G^{(N)}_\xi(z) = (z-1)\, p^{(N)}_{\delta_{i_m}}(1)\, G^{(N-1)}_{\xi-\delta_{i_m}}(z) + G^{(N)}_{\xi-\delta_{i_m}}(z) \tag{60}$$

Clearly

$$G_{\delta_i}^{(N)}(z) = \frac{i}{N+1} + \Big(1 - \frac{i}{N+1}\Big) z \tag{61}$$

Thus for the exclusion process the probability generating function for the number of particles absorbed at zero can be computed by iterating (60). For $m = 2$ and $i < j$ the recurrence (60) gives

$$G_{\delta_i+\delta_j}^{(N)}(z) \;=\; (z-1) p_{\delta_j}^{(N)}(1) G_{\delta_i}^{(N-1)}(z) + G_{\delta_i}^{(N)}(z) \tag{62}$$

Using (61) we get

$$\begin{aligned} G_{\delta_i+\delta_j}^{(N)}(z) &= (z-1)\Big(1 - \frac{j}{N+1}\Big)\Big[\frac{i}{N} + \Big(1 - \frac{i}{N}\Big) z\Big] \\ &+ \frac{i}{N+1} + \Big(1 - \frac{i}{N+1}\Big) z \end{aligned} \tag{63}$$

and one can check that this expression satisfies (53) with $m = 2$, i.e.

$$(1-z)\frac{d}{dz} G_{\delta_i+\delta_j}^{(N)}(z) + 2G_{\delta_i+\delta_j}^{(N)}(z) = G_{\delta_i}^{(N)}(z) + G_{\delta_j}^{(N)}(z) \tag{64}$$

For $m = 3$ and $i < j < k$ the recurrence (60) gives

$$G_{\delta_i+\delta_j+\delta_k}^{(N)}(z) \;=\; (z-1) p_{\delta_k}^{(N)}(1) G_{\delta_i+\delta_j}^{(N-1)}(z) + G_{\delta_i+\delta_j}^{(N)}(z) \tag{65}$$

Using (63) we get

$$\begin{aligned} G_{\delta_i+\delta_j+\delta_k}^{(N)}(z) &= (z-1)^2\Big(1 - \frac{k}{N+1}\Big)\Big(1 - \frac{j}{N}\Big)\Big[\frac{i}{N-1} + \Big(1 - \frac{i}{N-1}\Big) z\Big] \\ &+ (z-1)\Big(2 - \frac{k+j}{N+1}\Big)\Big[\frac{i}{N} + \Big(1 - \frac{i}{N}\Big) z\Big] \\ &+ \frac{i}{N+1} + \Big(1 - \frac{i}{N+1}\Big) z \end{aligned} \tag{66}$$

One can check that this expression satisfies (53) with $m = 3$, i.e.

$$(1-z)\frac{d}{dz} G_{\delta_i+\delta_j+\delta_k}^{(N)}(z) + 3G_{\delta_i+\delta_j+\delta_k}^{(N)}(z) = G_{\delta_i+\delta_j}^{(N)}(z) + G_{\delta_j+\delta_k}^{(N)}(z) + G_{\delta_k+\delta_i}^{(N)}(z) \tag{67}$$

6.3 Inclusion

For the inclusion process we can verify (53) with $m = 2$ using the results in [4]. Writing out

$$G_{\delta_i+\delta_j}(z) = p_{\delta_i+\delta_j}(0) + z p_{\delta_i+\delta_j}(1) + z^2 p_{\delta_i+\delta_j}(2) \tag{68}$$

we see that (53) is equivalent to

$$2p_{\delta_i+\delta_j}(0) + p_{\delta_i+\delta_j}(1) = \frac{i+j}{N+1} \tag{69}$$

From Eq. (5.6) in [4] giving the two-point correlation function for the Brownian Energy process, and using the fact that the inclusion process is self-dual, we can read off the absorption probabilities:

$$p_{\delta_i+\delta_j}(0) = \frac{i(2+j)}{(N+1)(N+3)} \tag{70}$$

and

$$p_{\delta_i+\delta_j}(1) = 1 - \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right) - \frac{i(2+j)}{(N+1)(N+3)} \tag{71}$$

Thus equation (69) is verified.

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