Orthogonal dualities of Markov processes and unitary symmetries

Gioia Carinci* Chiara Franceschini† Cristian Giardinà†

Wolter Groenevelt* Frank Redig*

December 21, 2018

Abstract

We study self-duality for interacting particle systems, where the particles move as continuous time random walkers having either exclusion interaction or inclusion interaction. We show that orthogonal self-dualities arise from unitary symmetries of the Markov generator. For these symmetries we provide two equivalent expressions that are related by the Baker-Campbell-Hausdorff formula. The first expression is the exponential of an anti Hermitian operator and thus is unitary by inspection; the second expression is factorized into three terms and is proved to be unitary by using generating functions. The factorized form is also obtained by using an independent approach based on scalar products, which is a new method of independent interest that we introduce to derive (bi)orthogonal duality functions from non-orthogonal duality functions.

*Technische Universiteit Delft, DIAM, P.O. Box 5031, 2600 GA Delft, The Netherlands.

†University of Modena and Reggio Emilia, FIM, via G. Campi 213/b, 41125 Modena, Italy.

1 Introduction

In a series of previous works, dualities that are *orthogonal* in an appropriate Hilbert space have been derived for a class of interacting particle systems with Lie-algebraic structure. This class includes several well-known processes, for instance the generalized exclusion processes [17, 23], the inclusion process [12], as well as independent random walkers [7]. These orthogonal dualities were identified as classical orthogonal polynomials in [8] by using the structural properties of those polynomials (recurrence relation and raising/lowering operators). In [20] the approach of generating functions was used instead, by which non-polynomial orthogonal dualities (provided by some other special functions, e.g. Bessel functions) were also found. Orthogonal duality functions can also be explained using representation theory: they can be understood as the intertwiner between two unitarily equivalent representations of a Lie algebra [9, 13].

Often the duality property of a Markov process can be related to the existence of some (hidden) symmetries of the Markov generator, i.e. operators commuting with the generator of the Markov process [10, 11]. This occurs for instance when the process has a reversible measure. In this context detailed balance can be interpreted as a trivial duality, and by acting with a symmetry of the generator one obtains a non-trivial duality. A natural question that arises is thus what type of symmetries lead to orthogonal dualities. In this paper we show that those symmetries have to be unitary and we single out the general expression that they must have.

The organization of this paper is as follows. In section 2 we give an overview of the main tools required to construct the setting. In 2.1 we recall the concept of (self-)duality between Markov processes and we introduce the notion of equivalence between (self-)duality functions. In 2.2 we introduce three algebras ($\mathfrak{su}(2)$ algebra, $\mathfrak{su}(1,1)$ algebra and the Heisenberg algebra) and the associated Markov processes that turn out to be interacting particle systems. In 2.3 we recall from [11] a general scheme to construct duality functions for Markov processes whose generator has an algebraic structure. In this approach there is a one-to-one correspondence between self-duality functions and symmetries of the Markov generator. In section 3.1, by using this connection between duality functions and symmetries we present the first main result of this paper. Namely, in Theorem 9 we provide the expression for the most general unitary symmetry that will then yield orthogonal duality functions. We also identify the special values of the parameters appearing in these symmetries for which the duality functions are orthogonal polynomials. The proof of Theorem 9 is contained in section 3.2. In section 3.3 we provide a second expression for these unitary symmetries: it is a factorized expression for function of the algebra generators that we show to be connected to the previous expression via the Baker–Campbell–Hausdorff formula. In section 4 we introduce a novel independent procedure to obtain orthogonal duality functions. This new method rely on the use of a scalar product in an Hilbert space. In 4.1 we prove that the scalar product of two duality functions is again a new duality function and in section 4.2 we show that these new duality functions are biorthogonal by construction. We apply this technique in section 4.3: for the interacting particle systems considered in this paper by manipulation of the biorthogonal relation we get an orthogonal relation.

The literature on stochastic duality for Markov processes is extremely vast. For the reader convenience we recall [14, 15, 19, 22, 24] for applications to non-equilibrium statistical physics, [3, 18] for duality in population models and [2, 5, 6] to study singular stochastic PDE via duality. Lastly, in [1] orthogonal duality is used to prove a Boltzman-Gibbs principle where several simplifications occur as a consequence of the fact that the duality functions constitute an orthogonal basis for the Hilbert space.

2 Preliminaries

We start by recalling the definition of stochastic duality for two processes and introducing the algebras and the interacting particle system (IPS) of interest. Our goal is to describe a constructive technique, in which self-duality functions arise from both the symmetric approach of section 2.3 as well as from the inner product approach described in section 4.

2.1 Stochastic duality

Definition 1 (Markov duality definitions.) *Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two continuous time Markov processes with state spaces $\mathscr{S}$ and $\mathscr{S}^{dual}$ and generators L and L^{dual} respectively. We say that Y is* dual *to X with duality function $D : \mathscr{S} \times \mathscr{S}^{dual} \longmapsto \mathbb{R}$ if*

$$\mathbb{E}_x[D(X_t, y)] = \mathbb{E}_y[D(x, Y_t)] \ , \tag{1}$$

for all $(x, y) \in \mathscr{S} \times \mathscr{S}^{dual}$ and $t \geq 0$. If X and Y are two independent copies of the same process, we say that Y is self-dual *with self-duality function D. Duality can also be regarded at the level of the processes generators. We say that L^{dual} is* dual *to L with duality function $D : \mathscr{S} \times \mathscr{S}^{dual} \longmapsto \mathbb{R}$ if*

$$[LD(\cdot, y)](x) = [L^{dual}D(x, \cdot)](y) \ . \tag{2}$$

if $L = L^{dual}$ we have self-duality.

Note that self-duality can always be thought as a special case of duality where the dual process is an independent copy of the first one. The simplification of self-duality for IPS typically arises from the fact that the copy process reduces to only a finite number of variables.

Countable state space. If the original process $(X_t)_{t\geq 0}$ and the dual process $(Y_t)_{t\geq 0}$ are Markov processes with countable state space $\mathscr{S}$ and $\mathscr{S}^{dual}$ resp., then the duality relation is equivalent to

$$\sum_{x'\in\mathscr{S}} L(x, x')D(x', y) = \sum_{y'\in\mathscr{S}} L^{dual}(y, y')D(x, y') = \sum_{y'\in\mathscr{S}} (L^{dual})^T(y', y)D(x, y') \tag{3}$$

where L^T denotes the transposition of the generator L. Generators are treated like (eventually infinite) matrices and in matrix notation the identity (3) becomes

$$LD = D(L^{dual})^T \ . \tag{4}$$

If $L^{dual} = L$ we obtain the corresponding identities for self-duality. In this context, the generator L is given by a matrix known as *rate matrix* such that

$$L(x, y) \geq 0 \qquad \text{for} \qquad x \neq y \qquad \text{and} \qquad \sum_y L(x, y) = 0 \ .$$

We say that the process jumps from x to y with *rate* $L(x, y)$.

Definition 2 (Duality functions in product form and single site duality functions.) *The duality functions we will present turn out to be of the following product structure*

$$D(\boldsymbol{x}, \boldsymbol{y}) = \prod_i d(x_i, y_i) \ .$$

The function inside the product will be regarded as single site duality functions and the subscript i removed.

Throughtout the paper we will work with duality functions of this structure and so we will only consider the single site.

Lemma 3 (Notion of equivalence for duality functions.) *If $D(x,y)$ is a duality function between two processes and the function $c : \mathscr{S} \times \mathscr{S}^{dual} \longrightarrow \mathbb{R}$ is constant under the dynamics of the two processes then $D_c(x,y) = c(x,y)D(x,y)$ is also a duality function. We will refer to D and D_c as equivalent duality functions.*

For example, in the context of the processes we are interested in, we will see that the dynamics conserves the total number of particles and dual particles, i.e. $\sum_i x_i = \sum_i n_i$ is conserved. As a consequence of this we can always choose a self-duality function up to a multiplicative factor in terms of the total number of particles. For example, if

$$D(x,y) = \prod_{i=1}^{n} d(x_i, y_i)$$

is a self-duality function, then for constants c and b, the function

$$D_{b,c}(x,y) = \prod_{i=1}^{n} b^{x_i} c^{y_i} d(x_i, y_i)$$

is again a self-duality. This can easily be checked using Definition 1. Indeed,

$$\begin{aligned}
\mathbb{E}_x\left(D_{b,c}(X(t),y)\right) &= \mathbb{E}_x\left(\prod_{i=1}^{n} b^{X_i(t)} c^{y_i} d(X_i(t), y_i)\right) = b^{\sum_i x_i} c^{\sum_i y_i} \mathbb{E}_x\left(D(X(t),y)\right) \\
&= b^{\sum_i x_i} c^{\sum_i y_i} \mathbb{E}_x\left(D(x,Y(t))\right) = \mathbb{E}_y\left(\prod_{i=1}^{n} b^{x_i} c^{Y_i(t)} d(x_i, Y_i(t))\right) \\
&= \mathbb{E}_y\left(D_{b,c}(x,Y(t))\right) \ .
\end{aligned}$$

Our examples are all such that $b = 1$ and so we omit it.

2.2 Algebras and IPS

In the next three sections we introduce three algebras with three IPS, each one corresponding to one of the three algebra. In particular, the probability measure that define the $*-$ structure of the algebra turns out to be the reversible measure of the particle process associated to that algebra. Here we denote by $\mathscr{F}(\mathscr{S})$ the space of real-valued functions on $\mathscr{S}$, with countable $\mathscr{S}$.

2.2.1 The Lie algebra $\mathfrak{su}(1,1)$ and symmetric inclusion process, SIP(k)

Generators of the *dual* Lie algebra $\mathfrak{su}(1,1)$ are K^0, K^+ and K^-. They satisfy

$$[K^0, K^{\pm}] = \mp K^{\pm} \qquad \text{and} \qquad [K^+, K^-] = 2K^0 \ . \tag{5}$$

The action of the three generators on functions f in $\mathscr{F}(\mathbb{N})$ is given by

$$\begin{cases} (K^+ f)(x) & := (2k+x) f(x+1) \\ (K^- f)(x) & := x f(x-1) \\ \left(K^0 f\right)(x) & := (x+k) f(x) \end{cases} \tag{6}$$

where $f(-1) = 0$. Equipping this setting with the inner product

$$\langle f, g \rangle_{w_{p,k}} = \sum_x f(x)g(x)w_{p,k}(x) \ , \qquad w_{p,k}(x) = \frac{\Gamma(2k+x)}{x!\Gamma(2k)} p^x (1-p)^{2k} \tag{7}$$

leads to the $*-$ structure

$$(K^0)^* = K^0, \qquad (K^+)^* = \frac{1}{p} K^-, \qquad (K^-)^* = pK^+ \ .$$

The Casimir element is

$$\Omega = 2(K^0)^2 - K^+K^- - K^-K^+$$

which is self-adjoint and it commutes with every element of the algebra.

The process associated with this algebra is the symmetric inclusion process, described below. The SIP($2k$) is a family of interacting particles processes labeled by the parameter $k > 0$ and that can be defined on a generic graph $G(V, E)$. The state space is unbounded so that each site can have an arbitrary number of particles. The SIP($2k$) generator is

$$L^{SIP(2k)} = \sum_{\substack{1 \le i < l \le |V| \\ (i,l) \in E}} w_{i,l} L_{i,l}^{SIP(2k)} \ , \tag{8}$$

$$L_{i,l}^{SIP(2k)} f(\mathbf{x}) = x_i(2k + x_l) \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + x_l(2k + x_i) \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right] \ ,$$

here $w_{i,l} > 0$, $\mathbf{x}^{i,l}$ denotes the particle configuration obtained from the configuration $\mathbf{x}$ by moving one particle from site i to site l, i.e. $\mathbf{x}^{i,l} = \mathbf{x} - \delta_i + \delta_l$ and so the dynamic conserves the total number of particles. Clearly, the action of the generator involves only two connected sites and it can be produced with the representation in system (6) via the expression of the coproduct of the Casimir Ω. Recall that the coproduct is an algebra homomorphism denoted by Δ and defined via the tensor product as

$$\Delta(X) = 1 \otimes X + X \otimes 1 \ ,$$

for the Lie algebra element X. In particular, one can verify that for the couple of sites (i, l) the generator of the SIP($2k$) on two sites using the generators of the $\mathfrak{su}(1, 1)$ Lie algebra

$$L_{i,l}^{SIP(2k)} = K_i^+ K_l^- + K_i^- K_l^+ - 2K_i^0 K_l^0 + 2k^2 = -\Delta(\Omega) + 2k^2 \ .$$

Last, the reversible measure of the SIP($2k$) process is given by the homogeneous product measure with marginals the Negative Binomial distributions with parameters $2k > 0$ and $0 < p < 1$, i.e. with probability mass function $w_{p,k}$ of equation (7).

2.2.2 The Lie algebra $\mathfrak{su}(2)$ and symmetric exclusion process, SEP($2j$)

Generators of the *dual* $\mathfrak{su}(2)$ Lie algebra are J^0, J^+ and J^- which satisfy the following commutation relations

$$[J^0, J^\pm] = \mp J^\pm \qquad \text{and} \qquad [J^+, J^-] = -2J^0 \ . \tag{9}$$

A representation of these three generators on functions f in $\mathscr{F}(\{0, 1, \ldots, 2j\})$ is given by

$$\begin{cases} \left(J^+ f\right)(x) & := (2j - x) f(x+1) \\ \left(J^- f\right)(x) & := x f(x-1) \\ \left(J^0 f\right)(x) & := (x - j) f(x) \end{cases} \tag{10}$$

where $f(-1) = f(2j+1) = 0$. Defining the inner product

$$\langle f, g \rangle_{w_{p,j}} = \sum_x f(x) g(x) w_{p,j}(x) \ , \qquad w_{p,j}(x) = \binom{2j}{x} \left(\frac{p}{1-p} \right)^x (1-p)^{2j} \tag{11}$$

the $*-$ structure is given by $(J^0)^* = J^0$, $(J^+)^* = \dfrac{1-p}{p} J^-$ and $(J^-)^* = \dfrac{p}{1-p} J^+$.
The Casimir element is

$$\Omega = 2(J^0)^2 + J^+ J^- + J^- J^+$$

which is central and self-adjoint. The process associated with this algebra is the exclusion process, defined below. The SEP($2j$) is a family of interacting particles processes labeled by the parameter $j \in \mathbb{N}/2$ and that can be defined on the same graph G, as before. Each site (vertex) of G can have at most $2j$ particles and the SEP($2j$) generator is

$$L^{SEP(2j)} = \sum_{\substack{1 \le i < l \le |V| \\ (i,l) \in E}} w_{i,l} L_{i,l}^{SEP(2j)} \ , \tag{12}$$

$$L_{i,l}^{SEP(2j)} f(\mathbf{x}) = x_i (2j - x_l) \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + (2j - x_i) x_l \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right] \ .$$

As before we can write the generator of the SEP($2j$) in two sites using the generator of the $\mathfrak{su}(2)$ algebra

$$L_{i,l}^{SEP(2j)} = J_i^+ J_l^- + J_i^- J_l^+ + 2 J_i^0 J_l^0 - 2j^2 = \Delta(\Omega) - 2j^2 \ .$$

Last, the reversible measure of the SEP($2j$) process is given by the homogeneous product measure with marginals the Binomial distribution with parameters $2j > 0$ and $p \in (0,1)$, i.e. with probability mass function $w_{p,j}$ of equation (11).

2.2.3 The Heisenberg algebra and independent random walkers (IRW)

The dual Heisenberg algebra is the Lie algebra with generators a and $a^\dagger$ such that

$$[a, a^\dagger] = -1 \ . \tag{13}$$

The Heisenberg algebra has a representation on $\mathscr{F}(\mathbb{N})$ such that

$$\begin{cases} \left(a^\dagger f \right)(x) = f(x+1) \\ \left(a f \right)(x) = x f(x-1) \end{cases} \tag{14}$$

Consider the inner product

$$\langle f, g \rangle_{w_p} = \sum_x f(x) g(x) w_p(x) \ , \qquad w_p(x) = \frac{p^x}{x!} e^{-p} \tag{15}$$

the representation above has $*-$ structure given by

$$a^* = p a^\dagger \quad \text{and} \quad \left(a^\dagger \right)^* = \frac{1}{p} a \ .$$

No such element as the Casimir is available with the Heisenberg algebra. The process associated within this algebra is the process of independent random walkers (IRW). They are defined in the usual setting, the process consists of independent particles that perform a symmetric continuous time random walk at rate 1 on the graph G. The generator is given by

$$\begin{aligned} L^{IRW} &= \sum_{\substack{1\le i<l\le N \\ (i,l)\in E}} w_{i,l} L_{i,l}^{IRW} \ , \\ L_{i,l}^{IRW} f(\mathbf{x}) &= x_i \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + x_l \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right] \ . \end{aligned} \tag{16}$$

The reversible invariant measure is provided by a homogeneous product of Poisson distributions with parameter $p > 0$, i.e., with probability mass function w_p of equation (15).

2.3 Self-dualities via symmetries: general approach and classical self-dualities

A general scheme for constructing self-dualities of continuous time Markov processes whose generator has a symmetry S, i.e. an operator commuting with its generator has been first proposed in [11]. We recall some known results in terms of trivial and classical self-duality functions and we show the corresponding original results for the discrete orthogonal polynomials found in [8, 20] . By construction, we are guaranteed that the functions we find via symmetries are self-dual, but not orthogonal. However, orthogonality can be inferred by proving that the symmetry is unitary. This orthogonality task is also address in section 4 where we show that biorthogonality can be achieved by construction. We will use this technique to find discrete orthogonal polynimoals as self-duality functions for our processes. Remind that, since our processes are defined on a countable state space $\mathscr{S}$, we can work with the notion of duality in matrix notation, namely equation (4).

Definition 4 *Let A and B be two matrices having the same dimension. We say that A is a symmetry of B if A commutes with B, i.e.*

$$[A, B] = AB - BA = 0 \ .$$

The main idea is that self-duality (in the context of Markov process with countable state space) can be recovered starting from a *trivial duality* which is based on the reversible measures of the processes. Then the action of a symmetry of the model on this trivial self-duality give rise into a non-trivial one. The following results, whose proof can be found in [11] formalize this idea.

Theorem 5 (Symmetries and self-duality.) *Let d be a self-duality function of the generator L and let S be a symmetry of L, then $D = Sd$ is again a self-duality function for L.*

If there is a description on the process generator in terms of a Lie algebra, then symmetries can be constructed using this algebraic structure. The two main elements of Theorem 5 are the initial self-duality d and the symmetry operator S. In general, if the process has a reversible measure the self-duality d can easily be found starting from the reversibility.

Lemma 6 (Diagonal self-duality and reversibility.) *If the process associated to generator L has reversible measure μ, then the diagonal self-duality functions are of the form*

$$d(x, y) = \frac{\delta_{x,y}}{\mu(x)} \quad \textit{where } x, y \in \mathscr{S} \ .$$

We refer to these diagonal self-duality functions as trivial or "cheap" self-duality functions. The next lemma summarize the cheap self-dualities for our three processes: notice that, up to neglectable factors, they are the inverse of their reversible measure.

Lemma 7 (Trivial self-duality functions.) *The processes of interests are self-dual with single site diagonal self-duality function given by*

$$D_p^{ch}(x,y) = \begin{cases} \dfrac{y!\Gamma(2k)}{\Gamma(2k+y)} p^{-y}\delta_{x,y} & \text{for the SIP(2k)} \\ \dfrac{(2j-y)!y!}{2j!}\left(\dfrac{1-p}{p}\right)^y \delta_{x,y} & \text{for the SEP(2j)} \\ \dfrac{y!}{p^y}\delta_{x,y} & \text{for the IRW} \end{cases} \tag{17}$$

We can now find several self-duality results applying the recipe of Theorem 5 starting with the trivial self-duality function. As symmetry, we use the exponential of the "lowering" operator of the suitable algebra: the fact that the generator of the process can be written as the coproduct of the Casimir element of the algebra, i.e. $\Delta(\Omega)$ guarantees that every algebra generator is a suitable symmetry. Moreover, we can provide the results for one site only and then generalize to a general graph of at least two sites. The following lemma shows how to find the so-called classical self-duality functions which have a lower triangular structure.

Proposition 8 (Classical self-duality functions and associated symmetries.) *The following results holds*

1. *The SIP(2k) is self-dual with single site self-duality function given by*

$$D_p^{cl}(x,y) := S\left(D_p^{ch}(\cdot,y)\right)(x) = \frac{x!}{(x-y)!}\frac{\Gamma(2k)}{\Gamma(2k+y)}p^{-y}\mathbf{1}_{\{y\le x\}} \quad \text{where} \quad S = e^{K^-} \ . \tag{18}$$

2. *The SEP(2j) is self-dual with single site self-duality function given by*

$$D_p^{cl}(x,y) := S\left(D_p^{ch}(\cdot,y)\right)(x) = \frac{x!}{(x-y)!}\frac{(2j-y)!}{2j!}\left(\frac{1-p}{p}\right)^y\mathbf{1}_{\{y\le x\}} \quad \text{where} \quad S = e^{J^-} \ . \tag{19}$$

3. *The IRW is self-dual with single site self-duality function given by*

$$D_p^{cl}(x,y) := S\left(D_p^{ch}(\cdot,y)\right)(x) = \frac{x!}{(x-y)!}\frac{1}{p^y}\mathbf{1}_{\{y\le x\}} \quad \text{where} \quad S = e^{a} \ . \tag{20}$$

Proof. We only consider the first item, the proof for the other two is similar. The proof that $D_p^{cl}(x,y)$ is self-duality is an immediate consequence of Theorem 5 since it is easy to see that K^- commutes with the Casimir Ω and so e^{K^-} commutes with every site of $D_p^{cl}(x,y)$ The second equality in (18) follows from a straightforward calculation. Indeed, acting with the symmetry S, we have

$$\begin{aligned} D^{cl}(x,y) = e^{K^-}\left(D_p^{ch}(\cdot,y)\right)(x) &= \sum_{i=0}^{\infty}\frac{(K^-)^i}{i!}\frac{y!\Gamma(2k)}{\Gamma(2k+y)}\left(\frac{1}{p}\right)^y\delta_{x,y} \\ &= \sum_{i=0}^{\infty}\frac{y!}{i!}\frac{\Gamma(2k)}{\Gamma(2k+y)}\frac{x!}{(x-i)!}\left(\frac{1}{p}\right)^y\mathbf{1}_{\{i\le x\}}\delta_{x-i,y} \\ &= \frac{x!}{(x-y)!}\frac{\Gamma(2k)}{\Gamma(2k+y)}\left(\frac{1}{p}\right)^y\mathbf{1}_{\{y\le x\}} \ . \end{aligned}$$

□

In virtue of Lemma 3 one can either neglect constants and factors that are constant under the dynamic of the process or, on the other hand, add convenient choice of these constant factors. In particular, in section 4, we will fix the value of these constants in a suitable way.

3 Orthogonal self-dualities and unitary symmetries

In what follows, we will relate the orthogonal polynomials with their hypergeometric functions. In general, the hypergeometric functions ${}_rF_s$ is defined as an infinite series

$$ {}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix};x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{x^k}{k!} $$

where $(a)_k$ denotes the Pochhammer symbol defined in terms of the Gamma function as

$$ (a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} \ . $$

Whenever one of the numerator parameters is a negative integer, the hypergeometric function ${}_rF_s$ turns into a finite sum, i.e. a polynomial. We define polynomials as in [16], in particular the following three discrete polynomilas: Meixner polynomials

$$ M(x,y;p) = {}_2F_1\left(\begin{matrix} -x,-y \\ 2k \end{matrix};1-\frac{1}{p}\right) \quad \text{for} \quad x,y\in\mathbb{N} \ , $$

Krawtchouk polynomials

$$ K(x,y;p) = {}_2F_1\left(\begin{matrix} -x,-y \\ -2j \end{matrix};\frac{1}{p}\right) \quad \text{for} \quad x,y = 0,1,\dots,2j \ , $$

and the Charlier polynomials

$$ C(x,y;p) = {}_2F_0\left(\begin{matrix} -x,-y \\ - \end{matrix};-\frac{1}{p}\right) \quad \text{for} \quad x,y\in\mathbb{N} \ . $$

3.1 Main result

In this section we explicitly determine the symmetries S, given in terms of the underling Lie algebra generators, which allow to retrieve the orthogonal polynomials. It is important to mention that, since we start from a (trivial) self-duality which is orthogonal with respect to the measure w, then the operator S that produce the orthogonal self-duality must be unitary. Recall that a unitary operator in $\mathsf{L}^2(\mathscr{S},w)$ is a linear operator such that

$$ UU^* = U^*U = I \ , $$

where U^* is the adjoint of U in $\mathsf{L}^2(\mathscr{S},w)$. As a consequence of this, we will have that U preserves the inner product of the Hilbert space $\mathsf{L}^2(\mathscr{S},w)$ and so the norm of the cheap self-duality function D^{ch} must be the same of the norm of the orthogonal self-duality function $D^{or} = SD^{ch}$

$$ \| D^{or} \|_w^2 = \langle SD^{ch}, SD^{ch}\rangle_w = \| D^{ch} \|_w^2 \ . $$

In the spirit of Proposition 8 we list the new orthogonal symmetries for the interacting particles systems.

Theorem 9 (Orthogonal self-duality functions and associated symmetries.) *The following results holds*

1. *For the SIP($2k$) we have that*

i) The symmetry

$$S_{\alpha,\beta} = \exp\left(\beta\left(-K^+ + \frac{1}{p}K^-\right)\right)\exp\left(i\alpha K^0\right) \tag{21}$$

is unitary for every choice of $\alpha, \beta \in \mathbb{R}$. *As a consequence* $S_{\alpha,\beta}\left(D_p^{ch}(x,\cdot)\right)(y)$ *are orthogonal (single site) self-duality functions in* $\mathsf{L}^2(w_{p,k})$ *with squared norm* $\| D_p^{ch} \|^2_{w_{p,k}}$.

ii) Choosing $\alpha = \hat{\alpha} = \pi$ *and* $\beta = \hat{\beta} = \sqrt{p}\,\mathrm{arctanh}\left(\sqrt{p}\right)$ *we get the Meixner polynomials up to a constant:* $D_p^{or}(x,y) := S_{\hat{\alpha},\hat{\beta}}\left(D_p^{ch}(x,\cdot)\right)(y) = (p-1)^k M(x,y;p)$.

2. For the SEP(2j) we have that

i) The symmetry

$$S_{\alpha,\beta} = \exp\left(\beta\left(-J^+ + \frac{1-p}{p}J^-\right)\right)\exp\left(i\alpha J^0\right) \tag{22}$$

is unitary for every choice of $\alpha, \beta \in \mathbb{R}$. *As a consequence* $S_{\alpha,\beta}\left(D_p^{ch}(x,\cdot)\right)(y)$ *are orthogonal (single site) self-duality functions in* $\mathsf{L}^2(w_{p,j})$ *with squared norm* $\| D_p^{ch} \|^2_{w_{p,j}}$.

ii) Choosing $\alpha = \hat{\alpha} = \pi$ *and* $\beta = \hat{\beta} = \sqrt{\frac{p}{1-p}}\arctan\left(\sqrt{\frac{p}{1-p}}\right)$ *we get the Krawtchouk polynomials up to a constant:* $D_p^{or}(x,y) := S_{\hat{\alpha},\hat{\beta}}\left(D_p^{ch}(x,\cdot)\right)(y) = (p-1)^j K(x,y;p)$.

3. For the IRW we have that

i) The symmetry

$$S_{\alpha,\beta} = \exp\left(\beta\left(-pa^\dagger + a\right)\right)\exp\left(i\alpha\, aa^\dagger\right) \tag{23}$$

is unitary for every choice of $\alpha, \beta \in \mathbb{R}$. *As a consequence* $S_{\alpha,\beta}\left(D_p^{ch}(x,\cdot)\right)(y)$ *are orthogonal (single site) self-duality functions in* $\mathsf{L}^2(w_p)$ *with squared norm* $\| D_p^{ch} \|^2_{w_p}$.

ii) Choosing $\alpha = \hat{\alpha} = \pi$ *and* $\beta = \hat{\beta} = 1$ *we get the Charlier polynomials up to a constant:* $D_p^{or}(x,y) := S_{\hat{\alpha},\hat{\beta}}\left(D_p^{ch}(x,\cdot)\right)(y) = e^{-\frac{p}{2}}C(x,y;p)$.

3.2 Proof of the main result

We need the following lemma to introduce the generating function and to compute the action of the algebra generators in order to prove Theorem 9. In particular we only consider the $\mathfrak{su}(1,1)$ algebra and the SIP$(2k)$ process but for the other two processes the idea is the same.

Definition 10 (Generating functions.) *We will always use the definition of generating function as in [16] (formula 9.10.11), i.e. the generating function* G *of* $g(y)$ *is defined as*

$$(Gg)(t) := \sum_{y=0}^{\infty} g(y)\frac{\Gamma(2k+y)}{y!\Gamma(2k)}t^y\ , \quad t \in \mathbb{R}\ . \tag{24}$$

The generating function of Meixner polynomials $M(x,y;p)$ (see [16]) is

$$\sum_{y=0}^{\infty} M(x,y;p)\frac{\Gamma(2k+y)}{y!\Gamma(2k)}t^y = \left(1-\frac{t}{p}\right)^x (1-t)^{-2k-x}\ . \tag{25}$$

Lemma 11 (Intertwinings of the $\mathfrak{su}(1,1)$ algebra generators.) *The following results hold*

1. $GK^{-}g(y) = \left(2k\ t + t^2 \frac{\partial}{\partial t}\right) Gg(t) =: \mathcal{K}^{-} Gg(t)$.

2. $GK^{+}g(y) = \left(\frac{\partial}{\partial t}\right) Gg(t) =: \mathcal{K}^{+} Gg(t)$.

3. $GK^{0}g(y) = \left(k + t\frac{\partial}{\partial t}\right) Gg(t) =: \mathcal{K}^{0} Gg(t)$.

Note that $\mathcal{K}^{-}$, $\mathcal{K}^{+}$ and $\mathcal{K}^{0}$ so defined, satisfy the dual $\mathfrak{su}(1,1)$ Lie algebra.

Proof.

$$\begin{aligned} GK^{-}g(y) &= \sum_{n=0}^{\infty} y g(y-1) \frac{\Gamma(2k+y)}{y!\Gamma(2k)} t^y \\ &= 2k\ t \sum_{y=0}^{\infty} g(y) \frac{\Gamma(2k+y)}{y!\Gamma(2k)} t^y + t^2 \sum_{y=0}^{\infty} g(y) \frac{\Gamma(2k+y)}{y!\Gamma(2k)} y t^{y-1} \\ &= \left(2k\ t + t^2 \frac{\partial}{\partial t}\right) Gg(t) \\ &= \mathcal{K}^{-} Gg(t)\ , \end{aligned}$$

this implicitly defines the operator $\mathcal{K}^{-}$ which acts on functions of the t variable as

$$\mathcal{K}^{-} := 2k\ t + t^2 \frac{\partial}{\partial t}\ .$$

Similarly,

$$GK^{+}g(y) = \sum_{y=0}^{\infty} (2k+y) g(y+1) \frac{\Gamma(2k+y)}{y!\Gamma(2k)} t^y = \sum_{y=0}^{\infty} g(y) \frac{\Gamma(2k+y)}{y!\Gamma(2k)} y t^{y-1} = \left(\frac{\partial}{\partial t}\right) Gg(t) = \mathcal{K}^{+} Gg(t)\ ,$$

so the operator $\mathcal{K}^{+}$ is a first derivative w. r. to t, defined as

$$\mathcal{K}^{+} f(t) := \frac{\partial f}{\partial t}(t)\ .$$

For K^0 we proceed in the same way

$$GK^{0}g(y) = \sum_{y=0}^{\infty} (k+y) g(y) \frac{\Gamma(2k+y)}{y!\Gamma(2k)} t^y = \left(k + t\frac{\partial}{\partial t}\right) Gg(t) = \mathcal{K}^{0} Gg(t)\ ,$$

and we infer that

$$\mathcal{K}^{0} f(t) := \left(k + t\frac{\partial}{\partial t}\right) f(t)\ .$$

Note that for all the above we have called $f(t) = (Gg(\cdot))\,(t)$.

□

Proof of Theorem 9. We will only give a proof for the first item as the other two follow a similar strategy. The first point of the first item regards the unitarity of $S_{\alpha,\beta}$ in $\mathsf{L}^2(w_{p,k})$, which is achieved if $(S_{\alpha,\beta})^* = (S_{\alpha,\beta})^{-1}$. Using the $*-$structure we have that $(S_{\alpha,\beta})^* = \exp\left(-i\alpha K^0\right)\exp\left(\beta\left(-\frac{1}{p}K^- + K^+\right)\right) = (S_{\alpha,\beta})^{-1}$ and so unitarity immediately follows. Unitary operators conserve the norm and so the norm of $S_{\alpha,\beta}D_p^{ch}(x,y)$ is the same as the norm of $D_p^{ch}(x,y)$ in $L^2(w_{p,k})$. In particular, the two squared norms are

$$\| S_{\alpha,\beta}D_p^{ch} \|^2_{w_{p,k}} = \| D_p^{ch} \|^2_{w_{p,k}} = \frac{y!\Gamma(2k)}{\Gamma(2k+y)}p^{-y}(1-p)^{2k} \, .$$

We show now the proof of the second point using a generating function approach. The idea is to show that the generating function of $D_p^{or} = S_{\hat{\alpha},\hat{\beta}}D_p^{ch}$ and the Meixner polynomials are the same, i.e.

$$G\left(S_{\hat{\alpha},\hat{\beta}}\left(D_p^{ch}(x,\cdot)\right)\right)(y) = G\Big((p-1)^k M(x,\cdot;p)\Big)(y) \tag{26}$$

and so using the generating function of Meixner polynomials in equation (25) one has that the r.h.s of equation (26) is $(p-1)^{-k}(1-t)^{-2k-x}\left(1-\frac{t}{p}\right)^x$. For the l.h.s. instead of computing $G\left(S_{\alpha,\beta}D_p^{ch}\right)(t)$ we use Lemma 11 to evaluate $\mathcal{S}_{\alpha,\beta}\left(GD_p^{ch}\right)(t)$, here $\mathcal{S}_{\alpha,\beta} = \exp\left(\beta\left(-\mathcal{K}^+ + \frac{1}{p}\mathcal{K}^-\right)\right)\exp\left(i\alpha\mathcal{K}^0\right)$, where $\mathcal{K}^+$, $\mathcal{K}^-$ and $\mathcal{K}^0$ are those in Lemma 11. In other words, we have to find the action of the operator $\mathcal{S}_{\alpha,\beta}$ on

$$\left(GD_p^{ch}\right)(t) = \sum_{y=0}^{\infty}\frac{y!\Gamma(2k)}{\Gamma(2k+y)}p^{-y}\delta_{x,y}\frac{\Gamma(2k+y)}{y!\Gamma(2k)}t^y = \left(\frac{t}{p}\right)^x \, . \tag{27}$$

The action of $\exp\left(i\alpha\mathcal{K}^0\right)$ on $f(t) = Gg(t)$ is

$$\left(e^{i\alpha}\right)^{\mathcal{K}^0} f(t) := G\left(\left(e^{i\alpha}\right)^{K^0} g\right)(t) = \sum_{y=0}^{\infty}\frac{\Gamma(2k+y)}{y!\Gamma(2k)}t^y(e^{i\alpha})^{y+k}g(y) = (e^{i\alpha})^k f\left(e^{i\alpha}t\right) \, .$$

Letting $\alpha = \hat{\alpha} = \pi$ one has

$$\exp\left(i\pi\mathcal{K}^0\right)f(t) = (-1)^k f(-t) \, . \tag{28}$$

To find the action of $\exp\left(\beta\left(-\mathcal{K}^+ + \frac{1}{p}\mathcal{K}^-\right)\right)$ we will solve a partial differential equation, whose solution $\psi(t,\beta)$ is the action of $\mathcal{S}_{\alpha,\beta}$ on function $f(t)$. Using Lemma 11, this is

$$\psi(t,\beta) = e^{\beta\left[\left(\frac{t^2}{p}-1\right)\frac{\partial}{\partial t} + \frac{2k}{p}\, t\right]}f(t) \tag{29}$$

with initial condition $\psi(t,0) = f(t)$. Deriving both sides of (29) w. r. to β we get a first order PDE for ψ:

$$\frac{\partial\psi}{\partial\beta} + \left(\frac{t^2}{p} - 1\right)\frac{\partial\psi}{\partial t} - \frac{2kt}{p}\psi = 0 \, . \tag{30}$$

To solve the PDE we use the method of characteristics: we consider ψ along the characteristic plane (τ, s), so that along a characteristic curve τ is constant and $\psi(t,\beta) = \psi(t(s),\beta(s))$. We then have

$$\frac{\partial\psi}{\partial s} = \frac{\partial\psi}{\partial\beta}\frac{\partial\beta}{\partial s} + \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial s} \, .$$

Comparing the above with the PDE in equation (30) we we just have to solve a system of three first order ODEs:

$$\begin{cases} \dfrac{\partial \beta}{\partial s} = 1 \\ \dfrac{\partial t}{\partial s} = \dfrac{p-t^2}{p} \\ \dfrac{\partial \psi}{\partial s} = \dfrac{2kt}{p}\psi \end{cases}$$

From the first equation we have immediately that $\beta = s$, while the second has solution

$$t(s) = \sqrt{p}\, \frac{\tanh(s/\sqrt{p}) + \tanh(c_1)}{1 + \tanh(s/\sqrt{p})\tanh(c_1)} \ .$$

using the initial condition $t(0) = \sqrt{p}\tanh(c_1) = \tau$ we get $c_1 = \operatorname{arctanh}\left(\tau/\sqrt{p}\right)$ and so

$$t(s) = \sqrt{p}\, \frac{\tau/\sqrt{p} + \tanh(s/\sqrt{p})}{1 + \tau/\sqrt{p}\tanh(s/\sqrt{p})} \ .$$

Substituting t in the last ODE we find that

$$\psi(s) = \left(\tau \sinh(s/\sqrt{p}) + \sqrt{p}\cosh(s/\sqrt{p})\right)^{2k} c_2 \ .$$

To find c_2 we use the initial condition in the characteristic plane, i.e. $\psi(0) = f(\tau) = p^k c_2$ so $c_2 = \frac{f(\tau)}{p^k}$ and so our solution in the (τ, s) plane is

$$\psi(\tau, s) = f(\tau)\left(\frac{\tau}{\sqrt{p}}\sinh(s/\sqrt{p}) + \cosh(s/\sqrt{p})\right)^{2k} \ .$$

In the (t, β) plane this becomes

$$\psi(t,\beta) = f\left(\sqrt{p}\, \frac{t - \sqrt{p}\ \tanh(\beta/\sqrt{p})}{\sqrt{p} - t\ \tanh(\beta/\sqrt{p})}\right)\left(-\frac{t}{\sqrt{p}}\ \sinh(\beta/\sqrt{p}) +\ \cosh(\beta/\sqrt{p})\right)^{-2k} \ .$$

Setting $\beta = \hat{\beta} = \operatorname{arctanh}\left(\sqrt{p}\right)\sqrt{p}$ the above expression simplifies to

$$e^{\hat{\beta}\left[\left(-1+\frac{t^2}{p}\right)\frac{\partial}{\partial t} + \frac{2k}{p}\ t\right]} f(t) = \left(\frac{1-t}{\sqrt{1-p}}\right)^{-2k} f\left(\frac{t-p}{1-t}\right) \ . \tag{31}$$

Equation (28) together with (31) finally gives

$$\mathcal{S}_{\hat{\alpha},\hat{\beta}} f(t) = (p-1)^k (1-t)^{-2k} f\left(\frac{p-t}{1-t}\right) \ . \tag{32}$$

Last, we need to set $f(t) = \left(GD_p^{ch}\right)(t) = \left(\dfrac{t}{p}\right)^x$ to finally get

$$\mathcal{S}_{\hat{\alpha},\hat{\beta}}\left(GD_p^{ch}\right)(t) = (p-1)^k (1-t)^{-2k-x}\left(1 - \frac{t}{p}\right)^x ,$$

which matches the generating function of the Meixner polynomials.

□

In the following section we give a different expression for the three unitary symmetries $S_{\hat{\alpha},\hat{\beta}}$ of Theorem 9.

3.3 Factorized symmetries

We now want to study the unitary symmetries that arises from the previous section. Since we do not know how to act with these symmetries on functions $f(x) \in \mathscr{F}(\mathbb{N})$, we wonder if a 'factorized' version of $S_{\hat{\alpha},\hat{\beta}}$ exists, i.e. if we can find a, b and c such that

$$S_{\hat{\alpha},\hat{\beta}} = e^{aK^-} e^{bK^0} e^{cK^+} \ .$$

The advantage of having a factorized symmetry is that one can directly compute its action on $f(x)$ (without passing via generating functions), even if, on the other hand, the unitary property is not an immediate consequence of this form. In the next section we will relate this factorized form to another symmetry.

Theorem 12 (Factorized unitary symmetries.) *The three orthogonal symmetries $S_{\hat{\alpha},\hat{\beta}}$ can also be written in a factorized version using the appropriate algebra generators.*

1. *The action of $S_{\hat{\alpha},\hat{\beta}}$ in equation (21) coincides with the action of $e^{K^-} e^{\log(p-1)K^0} e^{pK^+}$.*

2. *The action of $S_{\hat{\alpha},\hat{\beta}}$ in equation (22) coincides with the action of $e^{J^-} e^{\log\left(\frac{1}{p-1}\right)J^0} e^{\frac{p}{1-p}J^+}$.*

3. *The action of $S_{\hat{\beta}}$ in equation (23) coincides with the action of $e^{a} e^{-p/2+i\pi aa^\dagger} e^{pa^\dagger}$.*

Proof. We only show the first item as the other two have similar proofs; to do that we still use generating functions. To show that

$$e^{K^-} (p-1)^{K^0} e^{pK^+} g(y) = \exp\left(\hat{\beta}\left(-K^+ + \frac{1}{p}K^-\right)\right) \exp\left(i\hat{\alpha}K^0\right) g(y) \ , \tag{33}$$

we first consider the generating fucntion G on both sides and then flip the action of G with the one of the operators to get

$$e^{\mathcal{K}^-} (p-1)^{\mathcal{K}^0} e^{p\mathcal{K}^+} f(t) = \exp\left(\hat{\beta}\left(-\mathcal{K}^+ + \frac{1}{p}\mathcal{K}^-\right)\right) \exp\left(i\hat{\alpha}\mathcal{K}^0\right) f(t) \ , \tag{34}$$

where we called $f(t) = (G\ g)(t)$ and $\mathcal{K}^-$, $\mathcal{K}^0$ and $\mathcal{K}^+$ are those in Lemma 11. The r.h.s. of equation (34) has been evaluated in the proof of Theorem 9, equation (32) so we just need to find the action of $e^{\mathcal{K}^-}$, $(p-1)^{\mathcal{K}^0}$ and $e^{p\mathcal{K}^+}$. Clearly,

$$(p-1)^{\mathcal{K}^0} f(t) = (p-1)^k f\left(t\ (p-1)\right)$$

since

$$(p-1)^{\mathcal{K}^0} f(t) := G\left((p-1)^{K^0} g(y)\right) = \sum_{y=0}^{\infty} \frac{\Gamma(2k+y)}{y!\Gamma(2k)} t^y (p-1)^{y+k} g(y) = (p-1)^k f\left(t\,(p-1)\right) \ .$$

For $e^{\mathcal{K}^-}$ one can solve the associated PDE as in the proof of Theorem 9, or equivalently considering the limit as $p \to 0$ on both sides of equation (31) and using that $\lim_{p\to 0} \frac{\operatorname{arctanh}(\sqrt{p})}{\sqrt{p}} = 1$ leads to

$$e^{\mathcal{K}^-} f(t) = (1-t)^{-2k} f\left(\frac{t}{1-t}\right) \ .$$

Last, for $e^{p\mathcal{K}^+}$ we have that

$$\left(e^{p\mathcal{K}^+}f\right)(t) = e^{p\frac{\partial}{\partial t}}f(t) = f\left(t+p\right)$$

since the action of the first derivative is a shift. Acting on $f(t)$, we have

$$\begin{aligned}
e^{\mathcal{K}^-}\,(p-1)^{\mathcal{K}^0}\,e^{p\mathcal{K}^+}f(t) &= \\
e^{\mathcal{K}^-}\,(p-1)^{\mathcal{K}^0}\,f\left(t+p\right) &= \\
(p-1)^k e^{\mathcal{K}^-}\,f\left(t(p-1)+p\right) &= \\
(p-1)^k(1-t)^{-2k}f\left(\frac{t}{1-t}(p-1)+p\right) &= \\
(p-1)^k(1-t)^{-2k}f\left(\frac{p-t}{1-t}\right)&\;,
\end{aligned}$$

which matches the action of $S_{\hat{\alpha},\hat{\beta}}$ in equation (32).

□

Remark 13 (Baker-Campbell-Hausdorff formula for dual $\mathfrak{su}(1,1)$ algebra.) *The identity given in equation* (33) *can also be established as a consequence of the Baker-Campbell-Hausdorff formula for the* $\mathfrak{su}(1,1)$ *algebra, see [25] (formula* 24*b) adapted to the dual* $\mathfrak{su}(1,1)$ *algebra. In (*24*b) one has the following replacement* $L_+ = \frac{1}{\sqrt{p}}K^-$, $L_- = \sqrt{p}K^+$ *and* $L_0 = K^0$ *and in particular one has to set* $\tau = \operatorname{arctanh}(\sqrt{p})$ *and* $\alpha = \pi$.

The added value of having the factorized version of the symmetry $S_{\hat{\alpha},\hat{\beta}}$ is that one can immediately verify its action on the cheap duality $D_p^{ch}(x,y)$: via a straightforward computation one can produce the orthogonal polynomials of Theorem 9, as we show in the proposition below.

Proposition 14 (Direct computation of orthogonal polynomials.) *Acting with the factorized symmetry on the cheap self-duality function one gets the orthogonal self-duality function. In particular, for the SIP(2k) this is*

$$e^{K^-}e^{\log(p-1)K^0}e^{pK^+}\left(D_p^{ch}\left(x,\cdot\right)\right)(y) = D_p^{or}(x,y).$$

Proof. The proof follows a straightforward computation, see the Appendix.

□

4 Orthogonal self-duality via scalar products

In this section we first show how duality and self-duality function emerge as a consequence of what we call scalar product approach and which is introduced below. We then give some hypothesis to guarantee that such self-duality functions are biorthogonal. To conclude we implement this recently developed technique to find Meixner polynomials as orthogonal self-duality functions for the SIP$(2k)$, in a similar way one could find orthogonal self-dualities for SEP$(2j)$ and IRW.

4.1 Scalar product approach

In this section we present a new technique to approach duality: the naive idea is that the scalar product of two duality functions is still a duality function. We define the scalar product on some measure space $\mathsf{L}^2(\mathscr{S},\mu)$, in the usual way, i.e.

$$\langle f,g\rangle_\mu = \sum_{x\in\mathscr{S}} f(x)g(x)\mu(x)\ .$$

We will show that – in the setting of reversible processes – once two duality relations are available then it is possible to generate new different duality functions starting from the initial ones. Suppose we have three processes with generators L_1, L_2 and L_3 and state space $\mathscr{S}_1$, $\mathscr{S}_2$ and$\mathscr{S}_3$, respectively. In particular, assume that d_1 is a duality function for L_1 and L_2, while d_2 is a duality function for L_3 and L_2, i.e.

$$L_1 d_1(\cdot,y)(x) = L_2 d_1(x,\cdot)(y) \qquad \text{for} \quad (x,y)\in\mathscr{S}_1\times\mathscr{S}_2 \tag{35}$$

and

$$L_3 d_2(\cdot,y)(x) = L_2 d_2(x,\cdot)(y) \qquad \text{for} \quad (x,y)\in\mathscr{S}_3\times\mathscr{S}_2\ . \tag{36}$$

then the following proposition holds.

and the second process has reversible measure μ, then L_2 is self-adjoint on $\mathsf{L}^2(\mathscr{S},\mu)$.

Proposition 15 (New duality functions.) *If μ is a reversible measure for the generator L_2 and if equations* (35) *and* (36) *hold, then the function $D:\mathscr{S}_1\times\mathscr{S}_3\to\mathbb{R}$, given by*

$$D(x,y) = \langle d_1(x,\cdot), d_2(y,\cdot)\rangle_\mu \tag{37}$$

is a duality function for L_1 and L_3. If $L_1 = L_2 = L_3 = L$, then D is a new self-duality function for L.

Proof. For $i=1,2,3$, $L_{i,x}D(x,y)$ stands for $(L_i D(\cdot,y))(x)$ the action of L_i on the x variable of D. Then,

$$\begin{aligned}
L_{1,x}D(x,y) &= \langle L_{1,x}d_1(x,\cdot), d_2(y,\cdot)\rangle_\mu \\
&= \sum_z L_{2,z}d_1(x,z)d_2(y,z)\mu(z) \\
&= \sum_z d_1(x,z)L_{2,z}d_2(y,z)\mu(z) \\
&= \langle d_1(x,\cdot), L_{3,y}d_2(y,\cdot)\rangle_\mu = L_{3,y}D(x,y)\ ,
\end{aligned}$$

where we use duality of d_1 (resp. d_2) in the second (resp. fourth) equality and the self-adjointness of L_2 w. r. to μ.

□

A first application of the above proposition is shown in the example below, where we recover Laguerre polynomials as duality function between SIP$(2k)$ and BEP$(2k)$, which we do not introduce here, but it is well explained in [4] (section 2.2).

Example 16 (Duality via scalar product.) *A parametrized family of reversible measure for the SIP$(2k)$ process is*

$$\mu_p(z) = \frac{\Gamma(2k+z)}{\Gamma(2k)z!}p^z\ , \quad z\in\mathbb{N},\quad p\in(0,1)$$

and classical self-duality function D_p^{cl} for SIP(2k) is in equation (18)*, it will be our* $d_1(x,y)$*. The last ingredient we need is a duality function between BEP(2k) and SIP(2k), a well known in the literature ([4] see equation 4.9) is*

$$d_2(x,y) = \frac{x^y \Gamma(2k)}{\Gamma(2k+y)}(-1)^y \ , \quad x \in \mathbb{R}^+, \quad y \in \mathbb{N}.$$

In particular d_2 *is the one we need to obtain Laguerre polynomials. Proposition 15 assures us that* $D(x,y) = \langle d_2(x,\cdot), d_2(y,\cdot)\rangle_{\mu_p}$ *is a duality function between SIP(2k) and BEP(2k) and a straightforward computation shows that* D *is the closed form of the Laguerre polynomials. Indeed,*

$$\begin{aligned} D(x,y) = \langle d_2(x,\cdot), d_1(y,\cdot)\rangle_{\mu_p} &= \sum_{z=0}^{\infty} \frac{(-x)^z \Gamma(2k)}{\Gamma(2k+z)} \frac{y!}{(y-z)!} \frac{\Gamma(2k)}{\Gamma(2k+z)} p^{-z} \frac{\Gamma(2k+z)}{\Gamma(2k) z!} p^z \\ &= \sum_{z=0}^{y} \frac{(-x)^z}{z!} \frac{y!}{(y-z)!} \frac{\Gamma(2k)}{\Gamma(2k+z)} \\ &= {}_1F_1\left(\begin{matrix} -y \\ 2k \end{matrix} ; x \right) \quad \textit{for} \ \ y \in \mathbb{N} \ \ \textit{and} \ \ x \in \mathbb{R}^+ \ . \end{aligned}$$

We can apply Proposition 15 for the same generator, to construct the Meixner polynomials as SIP($2k$) self-duality functions.

Example 17 (Self-duality via scalar product.) *As for the previous Example 16, let* $\mu_p(z)$ *be the reversible measure for the SIP(2k) process. Consider now two classical self-duality functions* d_1 *and* d_2 *as in equation* (18)*. In particular, we are free to choose them without the constant, i.e.*

$$d_1(x,y) = d_2(x,y) = \frac{x!}{(x-y)!} \frac{\Gamma(2k)}{\Gamma(2k+y)} p^{-y} \ .$$

A simple computation shows that their scalar product in $\mathsf{L}^2(\mu_p)$ *is a Meixner polynomial. Indeed,*

$$\begin{aligned} D(x,y) = \langle d_2(x,\cdot), d_1(y,\cdot)\rangle_{\mu_p} &= \sum_{z=0}^{\infty} \frac{x!}{(x-z)!} \frac{y!}{(y-z)!} \left(\frac{\Gamma(2k)}{\Gamma(2k+z)} \right)^2 p^{-2z} \mathbf{1}_{z \le x} \mathbf{1}_{z \le y} \cdot \frac{\Gamma(2k+z)}{\Gamma(2k) z!} p^z \\ &= \sum_{z=0}^{x \wedge y} \frac{1}{z!} \frac{x!}{(x-z)!} \frac{y!}{(y-z)!} \frac{\Gamma(2k)}{\Gamma(2k+z)} p^{-z} \\ &= M(x,y;1-p) \quad \textit{for} \ \ x,y \in \mathbb{N} \ . \end{aligned}$$

The following proposition expands the result of Proposition 15 in the context of self-duality. It turns out that when two self-duality functions, d and D, are in a relation via a scalar product with a third function F, then, assuming d to be a basis for $\mathsf{L}^2(\mathscr{S},\mu)$, F must also be a self-duality function.

Proposition 18 (Basis and self-duality.) *Assume that* $\{x \mapsto d(x,n) \mid n \in \mathscr{S}\}$ *is a basis of self-duality functions for* $\mathsf{L}^2(\mathscr{S},\mu)$ *where* μ *is a reversible measure for the generator* L*. Let* $F = F(n,z)$ *be a function on* $\mathscr{S} \times \mathscr{S}$ *and define* D *by*

$$D(x,n) := \langle d(x,\cdot), F(n,\cdot)\rangle_\mu \ .$$

If D *is self-duality function, so is* F*.*

Proof. Using the short notation we have that

$$L_x D(x,n) = \langle L_x d(x,\cdot), F(n,\cdot)\rangle_\mu = \sum_{z\in\mathscr{S}} d(x,z) L_z F(n,z) \mu(z)$$

where we used that d is self-duality and that L is self-adjoint with respect to μ. On the other hand, since D is self-duality the above quantity must be equal to

$$L_n D(x,n) = \langle d(x,\cdot), L_n F(n,\cdot)\rangle_\mu = \sum_{z\in\mathscr{S}} d(x,z) L_n F(n,x)\mu(z) \ .$$

From the identity $L_x D(x,n) = L_n D(x,n)$, we have

$$\sum_{z\in\mathscr{S}} d(x,z)\left[L_z F(n,z) - L_n F(n,z)\right]\mu(z) = 0$$

and since d is a basis for $\mathsf{L}^2(\mathscr{S},\mu)$, necessarily $L_z F(n,z) - L_n F(n,z) = 0$, i.e. F is also a self-duality function for L.

□

4.2 Biorthogonal self-dualities

How does the orthogonal property play a role? Not all self-duality functions built with this method turn out to be orthogonal. However, there is a sort of stability with respect to this orthogonal property in the scalar product construction. More precisely, if we start with two biorthogonal self-duality functions the scalar product construction yields novel biorthogonal self-duality functions that may happen to be *equal* and therefore orthogonal.

To state the next proposition, we will use that the inverse of the reversible measure is a self-duality function as shown in Lemma 6.

Proposition 19 (Biorthogonal self-duality functions.) *Let μ_1 and μ_2 be two reversible measures and d_1, d_2 be two self-duality functions for the Markov process with generator L. Suppose that*

$$\langle d_1(x,\cdot), d_2(\cdot,n)\rangle_{\mu_1} = \frac{\delta_{x,n}}{\mu_2(n)} \qquad \textit{and} \qquad \langle d_2(x,\cdot), d_1(\cdot,n)\rangle_{\mu_2} = \frac{\delta_{x,n}}{\mu_1(n)} \ . \tag{38}$$

Then the functions

$$D(x,n) := \langle d_1(x,\cdot), d_1(n,\cdot)\rangle_{\mu_1} \qquad \widetilde{D}(x,n) := \langle d_2(\cdot,x), d_2(\cdot,n)\rangle_{\mu_1}.$$

are biorthogonal self-duality functions for L, i.e.

$$\langle D(\cdot,m), \widetilde{D}(\cdot,n)\rangle_{\mu_2} = \frac{\delta_{m,n}}{\mu_2(m)}.$$

In particular, if $D = \widetilde{D}$ we have the orthogonality relations for D.

Proof. From Proposition 15 we have that both D and $\widetilde{D}$ are self-duality functions since scalar product of self-dualities. Assuming now we can interchange the order of summation:

$$
\begin{aligned}
\langle D(\cdot, m), \widetilde{D}(\cdot, n)\rangle_{\mu_2} &= \sum_x D(x,m)\widetilde{D}(x,n)\mu_2(x) \\
&= \sum_x \left(\sum_y d_1(x,y)d_1(m,y)\mu_1(y)\right)\left(\sum_z d_2(z,x)d_2(z,n)\mu_1(z)\right)\mu_2(x) \\
&= \sum_{y,z} d_1(m,y)d_2(z,n)\mu_1(y)\mu_1(z)\sum_x d_2(z,x)d_1(x,y)\mu_2(x) \\
&= \sum_{y,z} \mu_1(y)\mu_1(z)d_1(m,y)d_2(z,n)\,\frac{\delta_{y,z}}{\mu_1(y)} \\
&= \sum_y d_1(m,y)d_2(y,n)\mu_1(y) = \frac{\delta_{m,n}}{\mu_2(m)}.
\end{aligned}
$$

□

We now implement this method to get the result below. Here we find Meixner polynomials as biorthogonal self-duality functions and with the aid of some hypergeometric functions transformation we find an orthogonal duality function.

4.3 From biorthogonal to orthogonality self-duality functions

According to Proposition 19 we need two duality functions d_1 and d_2 satisfying (38). For SIP($2k$) recall that

$$\mu_p(z) := \frac{\Gamma(2k+z)}{\Gamma(2k)z!}p^z$$

is the marginal of the (product) reversible (non normalized) measure, and

$$D^{cl}_{\frac{1}{\lambda}}(x,n) := \frac{x!\Gamma(2k)\lambda^y}{(x-y)!\Gamma(2k+y)}\,\mathbf{1}_{y\le x}$$

are the (single-site) classical self-duality functions. In the following we denote by $\langle\cdot,\cdot\rangle_p$ the scalar product w. r. to the non-normalized measure μ_p. We have the following lemma which we show for SIP($2k$).

Lemma 20 (Input self-duality functions.) *For any $p, q \in \mathbb{R}$ we have*

$$\langle D^{cl}_{-p}(x,\cdot), D^{cl}_{-q}(\cdot,n)\rangle_p = \frac{\delta_{x,n}}{\mu_q(x)}$$

where D^{cl} are the classical self-duality functions introduced in Proposition 8.

Proof. Note that $D^{cl}_{\frac{1}{\lambda}}(x,y) = 0$ if $y > x$. Then

$$\langle D^{cl}_{\frac{1}{\lambda}}(x,\cdot), D^{cl}_{\frac{1}{\alpha}}(\cdot,n)\rangle_p = \sum_{z=n}^{x} \frac{x!\Gamma(2k)\lambda^z}{(x-z)!\Gamma(2k+z)}\frac{z!\Gamma(2k)\alpha^n}{(z-n)!\Gamma(2k+n)}\frac{p^y\Gamma(2k+z)}{\Gamma(2k)z!}$$

which equals 0 if $x < n$. Suppose $x \ge n$, then shifting the summation index ($m = z - n$) gives

$$\langle D^{cl}_{\frac{1}{\lambda}}(x,\cdot), D^{cl}_{\frac{1}{\alpha}}(\cdot,n)\rangle_p = \frac{(p\lambda\alpha)^n\Gamma(2k)}{\Gamma(2k+n)}\sum_{m=0}^{x-n}\frac{x!}{(x-n-m)!m!}(\lambda p)^m\ .$$

Now use $(A)_{k+l} = (A)_k(A+k)_l$, i.e. $\dfrac{\Gamma(A+k+l)}{\Gamma(A)} = \dfrac{\Gamma(A+k)}{\Gamma(A)}\dfrac{\Gamma(A+k+l)}{\Gamma(A+k)}$ and the binomial theorem to obtain

$$\begin{aligned}\langle D^{cl}_{\frac{1}{\lambda}}(x,\cdot), D^{cl}_{\frac{1}{\alpha}}(\cdot,n)\rangle_p &= \frac{x!}{(x-n)!}\frac{\Gamma(2k)}{\Gamma(2k+n)}(p\lambda\alpha)^n \sum_{m=0}^{x-n} \frac{(x-n)!}{(x-n-m)!m!}(\lambda p)^m \\ &= \frac{x!}{(x-n)!}\frac{\Gamma(2k)}{\Gamma(2k+n)}(p\lambda\alpha)^n(1+\lambda p)^{x-n}\ .\end{aligned}$$

Setting $\lambda = -\frac{1}{p}$ and $\alpha = -\frac{1}{q}$ we get the result, i.e.

$$\langle D^{cl}_{-p}(x,\cdot), D^{cl}_{-q}(\cdot,n)\rangle_p = \frac{x!\Gamma(2k)}{\Gamma(2k+x)}\left(+\frac{1}{q}\right)^x \delta_{x,n} = \frac{\delta_{x,n}}{\mu_q(x)}\ .$$

□

Proposition 21 (From biorthogonal to orthogonality self-duality functions) *The self-duality functions*

$$D(x,n) = \langle D^{cl}_{-q}(x,\cdot), D^{cl}_{-q}(n,\cdot)\rangle_q$$

and

$$\widetilde{D}(x,n) = \langle D^{cl}_{-p}(\cdot,x), D^{cl}_{-p}(\cdot,n)\rangle_q$$

are biorthogonal w. r. to the stationary measure of their associated process. In details,

1. *For SIP(2k) we have that*

$$D(x,n) = {}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};\frac{1}{q}\right) \quad \textit{and} \quad \widetilde{D}(x,n) = \left(\frac{q}{p(1-q)}\right)^{n+x}(1-q)^{-2k}\,{}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};\frac{1}{q}\right)$$

   and they are biorthogonal w. r. to the measure w_p. In particular, for the choice

$$\frac{1}{q} = 1 - \frac{1}{p}$$

   we have

$$D(x,n) = M(x,n;p) \qquad \textit{and} \qquad \widetilde{D}(x,n) = (1-p)^{2k}\,D(x,n)$$

   so that the biorthogonality relation recovers the orthogonality relation of Meixner polynomials.

2. *For SEP(2j) we have that*

$$D(x,n) = {}_2F_1\left(\begin{matrix}-x,-n\\-2j\end{matrix};\frac{q-1}{q}\right) \quad \textit{and} \quad \widetilde{D}(x,n) = \left(\frac{q(p-1)}{p}\right)^{n+x}(1-q)^{-2j}\,{}_2F_1\left(\begin{matrix}-x,-n\\-2j\end{matrix};\frac{q-1}{q}\right)$$

   and they are biorthogonal w. r. to the measure w_p. In particular, for the choice

$$\frac{1}{q} = 1 - \frac{1}{p}$$

   we have

$$D(x,n) = K(x,n;p) \qquad \textit{and} \qquad \widetilde{D}(x,n) = (1-p)^{2j}\,D(x,n)$$

   so that the biorthogonality relation recovers the orthogonality relation of Krawtchouk polynomials.

3. For IRW we have that

$$D(x,n) = {}_2F_0\left(\begin{matrix}-x,-n\\-\end{matrix};\frac{1}{q}\right) \quad \text{and} \quad \widetilde{D}(x,n) = \left(-\frac{q}{p}\right)^{x+n} e^q\, {}_2F_0\left(\begin{matrix}-x,-n\\-\end{matrix};\frac{1}{q}\right)$$

and they are biorthogonal w. r. to the measure w_p. In particular, for the choice

$$q = -p$$

we have

$$D(x,n) = C(x,n;p) \qquad \text{and} \qquad \widetilde{D}(x,n) = e^{-p} D(x,n)$$

so that the biorthogonality relation recovers the orthogonality relation of Charlier polynomials.

Proof. As always we show the proof for the first item only. Now we apply Proposition 19 with

$$d_1(x,n) = D^{cl}_{-q}(x,n) \quad \text{and} \quad d_2(x,n) = D^{cl}_{-p}(x,n)$$

then, from Lemma (20), the conditions (38) are satisfied for $\mu_1 = \mu_q$ and $\mu_2 = \mu_p$. We have

$$D(x,n) = \langle D^{cl}_{-q}(x,\cdot), D^{cl}_{-q}(n,\cdot)\rangle_q = {}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};\frac{1}{q}\right) = M\left(x,n;\frac{q}{q-1}\right)$$

and

$$\begin{aligned}
\widetilde{D}(x,n) &= \langle D^{cl}_{-p}(\cdot,x), D^{cl}_{-p}(\cdot,n)\rangle_q \\
&= (-p)^{-x-n} \sum_{z=n}^{\infty} \frac{(2k)_z q^z}{z!} \frac{z!}{(z-x)!(2k)_x} \frac{z!}{(z-n)!(2k)_n} \\
&= \frac{(-p)^{-n-x} q^n}{(2k)_x (2k)_n} \sum_{m=0}^{\infty} \frac{(2k)_{m+n}(m+n)!}{m!(m+n-x)!}\, q^m \\
&= \frac{(-p)^{-n-x} q^n n!}{(2k)_x (n-x)!} \sum_{m=0}^{\infty} \frac{(2k+n)_m (n+1)_m}{m!(n-x+1)_m}\, q^m \\
&= \frac{(-p)^{-n-x} q^n n!}{(2k)_x (n-x)!}\, {}_2F_1\left(\begin{matrix}2k+n,n+1\\n-x+1\end{matrix};q\right).
\end{aligned}$$

By applying the following ${}_2F_1$-transformations [16, (2.2.6),(2.3.14),(2.2.6)], we obtain

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}2k+n,n+1\\n-x+1\end{matrix};q\right) &= (1-q)^{-n-2k}\, {}_2F_1\left(\begin{matrix}2k+n,-x\\n-x+1\end{matrix};\frac{q}{q-1}\right) \\
&= (1-q)^{-n-2k} \frac{(2k)_x}{(-n)_x}\, {}_2F_1\left(\begin{matrix}-x,2k+n\\2k\end{matrix};\frac{1}{1-q}\right) \\
&= (1-q)^{-n-x-2k} (-q)^x \frac{(2k)_x}{(-n)_x}\, {}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};\frac{1}{q}\right).
\end{aligned}$$

This gives

$$\widetilde{D}(x,n) = \left(-\frac{q}{p(1-q)}\right)^{n+x} (1-q)^{-2k}\, {}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};\frac{1}{q}\right)$$

then, for $\frac{1}{p} = 1 - \frac{1}{q}$ we get

$$\widetilde{D}(x,n) = (1-q)^{-2k}\, {}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};\frac{1}{q}\right) = (1-p)^{2k}\, {}_2F_1\left(\begin{matrix}-x,-n\\2k\end{matrix};1-\frac{1}{p}\right) = (1-p)^{2k} M(x,n;p).$$

□

Remark 22 *From the previous proposition we have that we can write the Meixner-duality function* $D(x,n) = M(x,n;p)$ *in two forms in terms of scalar product:*

$$D(x,n) = \langle D^{cl}_{\frac{p}{1-p}}(x,\cdot), D^{cl}_{\frac{p}{1-p}}(n,\cdot)\rangle_{\frac{p}{p-1}} = (1-p)^{-2k} \langle D^{cl}_{-p}(\cdot,x), D^{cl}_{-p}(\cdot,n)\rangle_{\frac{p}{p-1}} \,. \tag{39}$$

We now write $D^{cl}_{\frac{p}{1-p}}(x,y)$ and $D^{cl}_{-p}(y,x)$ as a symmetry acting on the cheap duality. This allows us to write both expressions for $D(x,n)$ in (39)via two symmetries (S_1 and S_2) acting on the cheap duality. Before doing that, we will use the following lemma to justify some equality in the computation below.

Lemma 23 (Duality of K^+ and K^- via the cheap duality function.) *the generators K^+ and K^- of the* $\mathfrak{su}(1,1)$ *algebra are dual via*

$$D^{ch}_1(x,y) = \frac{x!\Gamma(2k)}{\Gamma(2k+x)}\delta_{x,y} \,,$$

Proof.

$$\left(K^+ D^{ch}_1(\cdot,y)\right)(x) = \frac{y!\Gamma(2k)}{\Gamma(2k+y-1)} = \left(K^- D^{ch}_1(x,\cdot)\right)(y) \,.$$

□

As a consequence of the above relation, we have the following corollary.

Corollary 24 *The operators* e^{K^+} *and* e^{K^-} *are also in duality via* D^{ch}_1*. Moreover, we can choose parameter* α*,* β *on the exponentials and* λ *on* D^{ch}_1 *such that the relation*

$$\left(e^{\alpha\ K^-} D^{ch}_{\frac{1}{\lambda}}(\cdot,y)\right)(x) = \left(e^{\beta\ K^+} D^{ch}_{\frac{1}{\lambda}}(x,\cdot)\right)(y)$$

is always true for any α*,* β *and* $\lambda \in \mathbb{R}$ *that satisfy* $\alpha = \lambda\beta$*.*

To make notation simpler we write K^-_1 (resp. K^-_2) for the action of K^- on the first (resp. second) variable and same for K^+. Let's now investigate the two symmetries associated to the self-duality function in equation (39).

Proposition 25 (Two ways of expressing orthogonal polynomials.) *Let D be the self-duality function given by the two scalar products in equation* (39)*, then D can be written as a symmetry acting on* D^{ch}*. In particular, we have that*

$$D(x,n) = e^{K^-_1}\, e^{\frac{p}{p-1}K^+_1}\, D^{ch}_{\frac{p}{p-1}}(x,n) \tag{40}$$

for the first scalar product in (39)*, and*

$$D(x,n) = (1-p)^{-2k}\, e^{-pK^+_1} e^{\frac{1}{1-p}K^-_1}\, D^{ch}_{p(p-1)}(x,n) \tag{41}$$

for the second one.

Proof. For the first scalar product in (39) we have that, using the first item of Proposition 8

$$D(x,n) = \langle D^{cl}_{\frac{p}{1-p}}(x,\cdot), D^{cl}_{\frac{p}{1-p}}(n,\cdot)\rangle_{\frac{p}{p-1}} = \langle e^{K_1^-} D^{ch}_{\frac{p}{1-p}}(x,\cdot), e^{K_1^-} D^{ch}_{\frac{p}{1-p}}(n,\cdot)\rangle_{\frac{p}{p-1}} \ . \tag{42}$$

The action of $e^{K_1^-}$ only affect the x variable and it can be placed outside the scalar product, moreover by Corollary 24, the first scalar product in (42) is equal to

$$e^{K_1^-} \langle D^{ch}_{\frac{p}{1-p}}(x,\cdot), e^{\frac{p}{1-p}K_2^+} D^{ch}_{\frac{p}{1-p}}(n,\cdot)\rangle_{\frac{p}{p-1}} \ .$$

The adjoint of K^+ with respect to $\mu_{\frac{p}{p-1}}$ is $\frac{p-1}{p}K^-$ and the above quantity become

$$e^{K_1^-} \langle e^{-K_2^-} D^{ch}_{\frac{p}{1-p}}(x,\cdot), D^{ch}_{\frac{p}{1-p}}(n,\cdot)\rangle_{\frac{p}{p-1}} \ .$$

Using again Corollary 24 we finally get

$$e^{K_1^-} e^{\frac{p}{p-1}K_1^+} \langle D^{ch}_{\frac{p}{1-p}}(x,\cdot), D^{ch}_{\frac{p}{1-p}}(n,\cdot)\rangle_{\frac{p}{p-1}} = e^{K_1^-} e^{\frac{p}{p-1}K_1^+} D^{ch}_{\frac{p}{p-1}}(x,n) \ ,$$

where we computed the scalar product to get the symmetry in (40). In a similar fashion, for the second scalar product in (42) we have that

$$D(x,n) = (1-p)^{-2k} \langle D^{cl}_{-p}(\cdot,x), D^{cl}_{-p}(\cdot,n)\rangle_{\frac{p}{p-1}} = (1-p)^{-2k} \langle e^{K_2^-} D^{ch}_{-p}(x,\cdot), e^{K_2^-} D^{ch}_{-p}(n,\cdot)\rangle_{\frac{p}{p-1}}$$

by Corollary 24. Then, by considering the adjoint of K^- with respect to $\mu_{\frac{p}{p-1}}$, the above expression reads

$$(1-p)^{-2k} e^{-pK_1^+} \langle e^{\frac{p}{p-1}K_2^+} D^{ch}_{-p}(x,\cdot), D^{ch}_{-p}(n,\cdot)\rangle_{\frac{p}{p-1}}$$

and again, by Corollary 24 we have the symmetry in (41)

$$(1-p)^{-2k} e^{-pK_1^+} e^{\frac{1}{1-p}K_1^-} \langle D^{ch}_{-p}(x,\cdot), D^{ch}_{-p}(n,\cdot)\rangle_{\frac{p}{p-1}} = (1-p)^{-2k} e^{-pK_1^+} e^{\frac{1}{1-p}K_1^-} D^{ch}_{p(p-1)}(x,n) \ .$$

□

Remark 26 (A commutation relation for the exponentials.) *The expression for $D(x,n)$ in (41) can be written as an action on $D^{ch}_{p(p-1)}(x,n)$ as follows,*

$$D(x,n) = (1-p)^{-2k} e^{-pK_1^+} e^{\frac{1}{1-p}K_1^-} D^{ch}_{p(p-1)}(x,n) = e^{-pK_1^+} e^{\frac{1}{1-p}K_1^-} (1-p)^{-2K_1^0} D^{ch}_{\frac{p}{p-1}}(x,n) \ ,$$

comparing with (40) allows us to infer the following relation

$$e^{K^-} e^{\frac{p}{p-1}K^+} = e^{-pK^+} e^{\frac{1}{1-p}K^-} (1-p)^{-2K^0} \ . \tag{43}$$

Relation in (43) is found in [21] (Remark 3.2) adapted to the $\mathfrak{su}(1,1)$ Lie algebra generators.

We now do some consideration about the relations among the symmetries in equations (40) and (41) with the one obtained in Proposition 12. In order to compare their action, we first realize that

$$D^{ch}_{\frac{p}{p-1}}(x,n) = (p-1)^x D^{ch}_p(x,y) \ ,$$

so that their action on D^{ch}_p reads

$$(p-1)^k \ D(x,n) = S_1 D^{ch}_p(x,y) := e^{K^-} e^{\frac{p}{p-1}K^+} (p-1)^{K^0} D^{ch}_p(x,y)$$

and

$$(p-1)^k \ D(x,n) = S_2 D^{ch}_p(x,y) := e^{-pK^+} e^{\frac{1}{1-p}K^-} (p-1)^{-K^0} D^{ch}_p(x,y) \ .$$

It is easy to verify that both S_1 and S_2 are unitary in $w_{p,k}$ since the self-duality functions $(p-1)^k \ D(x,n)$ and $D^{ch}_p(x,y)$ have the same norm. One could also check it via the generating function approach.

Remark 27 (Equivalence of the symmetries.) *As a final remark we mention that the symmetry from Proposition 12 and S_1 are, as expected, the same. If we consider their action on $D^{ch}_p(x,y)$, then*

$$\left(e^{K^-}e^{\frac{p}{p-1}K^+}(p-1)^{K^0}D^{ch}_p(\cdot,y)\right)(x)=(p-1)^kM(x,y;p)$$

while

$$\left(e^{K^-}(p-1)^{K^0}e^{pK^+}D^{ch}_p(\cdot,y)\right)(x)=(p-1)^kM(x,y;p)$$

inferring the commutation relation between K^+ and K^0:

$$e^{\frac{p}{p-1}K^+}(p-1)^{K^0}=(p-1)^{K^0}e^{pK^+}\ .$$

5 Appendix

Proof of Proposition 14. We have to show that

$$e^{K^-}e^{\log(p-1)K^0}e^{pK^+}\left(D^{ch}_p(x,\cdot)\right)(y)=D^{or}_p(x,y),$$

where $D^{ch}_p(x,y)=\dfrac{y!\Gamma(2k)}{\Gamma(2k+y)}p^{-y}$ and $D^{or}_p(x,y)=(p-1)^k\,{}_2F_1\left({-x,-y \atop 2k};1-\frac{1}{p}\right)$.
We start by acting with the inner operator on the y variable of D^{ch}_p:

$$\begin{aligned}
&e^{K^-}e^{\log(p-1)K^0}e^{pK^+}\left(D^{ch}_p(x,\cdot)\right)(y)\\
&=e^{K^-}e^{\log(p-1)K^0}\sum_{i=0}^{\infty}\frac{p^i}{i!}\left(K^+\right)^i\frac{x!\Gamma(2k)}{\Gamma(2k+x)}(p)^{-x}\delta_{x,y}\\
&=e^{K^-}e^{\log(p-1)K^0}\sum_{i=0}^{\infty}\frac{p^i}{i!}\frac{\Gamma(2k+y+i)}{\Gamma(2k+y)}\frac{x!\Gamma(2k)}{\Gamma(2k+x)}p^{\cdot x}\delta_{x,y+i}
\end{aligned}$$

since the action of the i^{th} power of K^+ is $\left(K^+\right)^i f(y)=\frac{\Gamma(2k+y+i)}{\Gamma(2k+y)}f(y+i)$. The action of K^0 is diagonal and so we have

$$e^{K^-}e^{\log(p-1)K^0}p^{x-y}\frac{x!}{(x-y)!}\frac{\Gamma(2k)}{\Gamma(2k+y)}p^{-x}\mathbf{1}_{\{x\geq y\}}=e^{K^-}(p-1)^{y+k}\frac{x!}{(x-y)!}\frac{\Gamma(2k)}{\Gamma(2k+y)}\left(\frac{1}{p}\right)^y\mathbf{1}_{\{x\geq y\}}\ .$$

Finally we use the action of the i^{th} power of K^-, i.e. $\left(K^-\right)^i f(y)=\frac{y!}{(y-i)!}f(y-i)$.

$$\begin{aligned}
&(p-1)^k\sum_{i=0}^{\infty}\frac{\left(K^-\right)^i}{i!}\left(\frac{p-1}{p}\right)^y\frac{x!}{(x-y)!}\frac{\Gamma(2k)}{\Gamma(2k+y)}\mathbf{1}_{\{x\geq y\}}\\
&=(p-1)^k\sum_{i=0}^{\infty}\frac{1}{i!}\frac{y!}{(y-i)!}\left(\frac{p-1}{p}\right)^{y-i}\frac{x!}{(x-y+i)!}\frac{\Gamma(2k)}{\Gamma(2k+y-i)}\mathbf{1}_{\{x\geq y-i\}}\mathbf{1}_{\{i\leq y\}}\\
&=(p-1)^k\sum_{i=0\vee(y-x)}^{y}\frac{1}{i!}\frac{y!}{(y-i)!}\left(\frac{p-1}{p}\right)^{y-i}\frac{x!}{(x-y+i)!}\frac{\Gamma(2k)}{\Gamma(2k+y-i)}\\
&=(p-1)^k\sum_{s=0}^{x\wedge y}\frac{x!}{(x-s)!}\frac{y!}{(y-s)!}\frac{1}{s!}\left(\frac{p-1}{p}\right)^s\frac{\Gamma(2k)}{\Gamma(2k+s)}\\
&=(p-1)^kM(x,y;p)\\
&=D^{or}_p(x,y).
\end{aligned}$$

where we performed the change of variable $y-i=s$ at the end. Note that, up to the constant $(p-1)^k$, the last sum is the closed form of Meixner polynomials of parameter p and $2k$, variable x and degree y. Since the result is symmetric in x and y then the action on the x variable would produce the same result.

□

References

[1] M. Ayala, G. Carinci, F. Redig. Quantitative Boltzmann Gibbs principles via orthogonal polynomial duality. Journal of Statistical Physics 171.6, 980–999 (2018).

[2] A. Borodin, I. Corwin, T. Sasamoto. From duality to determinants for q-TASEP and ASEP. The Annals of Probability 42.6, 2314–2382 (2014).

[3] G. Carinci, C. Giardinà, C. Giberti, F. Redig. Dualities in population genetics: a fresh look with new dualities. Stochastic Processes and their Applications 125.3, 941–969 (2015).

[4] G. Carinci, C. Giardinà, C. Giberti, F. Redig. Duality for stochastic models of transport. Journal of Statistical Physics 152.4, 657–697 (2013).

[5] I. Corwin, P. Ghosal, H. Shen, T. Li-Cheng. Stochastic PDE Limit of the Six Vertex Model. Preprint arXiv:1803.08120 (2018).

[6] I. Corwin, H. Shen, L-C Tsai. ASEP(q,j) converges to the KPZ equation. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 54.2, 995–1012 (2018).

[7] A. De Masi, E. Presutti. *Mathematical methods for hydrodynamic limits.* Springer (2006).

[8] C. Franceschini, C. Giardinà. Stochastic Duality and Orthogonal Polynomials. Preprint arXiv:1701.09115 (2017).

[9] C. Franceschini, C. Giardinà, W. Groenevelt. Self-duality of Markov processes and intertwining functions. Math Phys Anal Geom 21: 29 (2018).

[10] C. Giardinà, J. Kurchan, F. Redig. Duality and exact correlations for a model of heat conduction. Journal of Mathematical Physics 48.3, 033301 (2007).

[11] C. Giardinà, J. Kurchan, F. Redig, K. Vafayi. Duality and hidden symmetries in interacting particle systems. Journal of Statistical Physics 135.1, 25–55 (2009).

[12] C. Giardinà, F. Redig, K. Vafayi. Correlation Inequalities for interacting particle systems with duality Journal of Statistical Physics Vol.141, 242–263 (2010).

[13] W. Groenevelt. Orthogonal stochastic duality functions from Lie algebra representations. Journal of Statistical Physics (2018).

[14] T. Imamura, T. Sasamoto. Current moments of 1D ASEP by duality. Journal of Statistical Physics 142.5, 919–930 (2011).

[15] C. Kipnis, C. Marchioro, E. Presutti. Heat flow in an exactly solvable model. Journal of Statistical Physics 27.1, 65–74 (1982).

[16] R. Koekoek, P.A. Lesky, R.F. Swarttouw. *Hypergeometric Orthogonal Polynomials and their* $q-$*Analogues*, Springer (2010).

[17] T. M. Liggett. *Interacting particles systems*. Springer (1985).

[18] M. Möhle. The concept of duality and applications to Markov processes arising in neutral population genetics models. Bernoulli 5.5, 761-–777 (1999).

[19] J. Ohkubo. On dualities for SSEP and ASEP with open boundary conditions. Journal of Physics A: Mathematical and Theoretical, (2017).

[20] F. Redig, F. Sau. Factorized duality, stationary product measures and generating functions. Journal of Statistical Physics 172.4, 980–1008 (2018).

[21] H. Rosengren. A new quantum algebraic interpretation of the Askey-Wilson polynomials. Contemporary Mathematics 254 371–394, AMS (2000).

[22] G.M. Schütz. Duality relations for asymmetric exclusion processes. Journal of Statistical Physics 86.5, 1265–1287 (1997).

[23] G.M. Schütz, S. Sandow. Non-Abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems. Physical Review E **49**, 2726 (1994).

[24] H. Spohn. Long range correlations for stochastic lattice gases in a non-equilibrium steady state. Journal of Physics A: Mathematical and General 16.18, 4275 (1983).

[25] D. R. Truax. Baker-Campbell-Hausdorff relations and unitarity of $SU(2)$ and $SU(1,1)$ squeeze operators. Physical Review D. 31.8, APS (1985).