Asymmetric stochastic transport models with $\mathscr{U}_q(\mathfrak{su}(1,1))$ symmetry

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Abstract

By using the algebraic construction outlined in [10], we introduce several Markov processes related to the $\mathscr{U}_q(\mathfrak{su}(1,1))$ quantum Lie algebra. These processes serve as asymmetric transport models and their algebraic structure easily allows to deduce duality properties of the systems. The results include: (a) the asymmetric version of the Inclusion Process, which is self-dual; (b) the diffusion limit of this process, which is a natural asymmetric analogue of the Brownian Energy Process and which turns out to have the symmetric Inclusion Process as a dual process; (c) the asymmetric analogue of the KMP Process, which also turns out to have a symmetric dual process. We give applications of the various duality relations by computing exponential moments of the current.

Contents

1 Introduction

1.1 Motivations

Exactly solvable stochastic systems out-of-equilibrium have received considerable attention in recent days [27, 16, 11, 24, 6, 5]. Often in the analysis of these models duality (or self-duality) is a crucial ingredient by which the study of n-point correlations is reduced to the study of n dual particles. For instance, the exact current statistics in the case of the asymmetric exclusion process is obtained by solving the dual particle dynamics via Bethe ansatz [26, 18, 4].

The duality property has algebraic roots, as was first noticed by Schütz and Sandow for symmetric exclusion processes [25], which is related to the classical Lie algebra $\mathfrak{su}(2)$. Next this symmetry approach was extended by Schütz [26] to the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(2))$ in a representattion of spin 1/2, thus providing self-duality of the asymmetric exclusion process. Recently Markov processes with the $\mathscr{U}_q(\mathfrak{su}(2))$ algebraic structure for higher spin value have been introduced and studied in [10]. This lead to a family of non-integrable asymmetric generalization of the partial exclusion process (see also [22]).

In [13, 14] the algebraic approach to duality has been extended by connecting duality functions to the algebra of operators commuting with the generator of the process. In particular for the models of heat conduction studied in [14] the underlying algebraic structure turned out to be $\mathscr{U}(\mathfrak{su}(1,1))$. This class is richer than its fermionic counterpart related to the classical Lie algebra $\mathscr{U}(\mathfrak{su}(2))$ which is at the root of processes of exclusion type. In particular, the classical Lie algebra $\mathscr{U}(\mathfrak{su}(1,1))$ has been shown to be related to a large class of symmetric processes, including: (a) an interacting particle system with attractive interactions (inclusion process [14, 15]); (b) interacting diffusion processes for heat conduction (Brownian energy process [14, 9]); (c) redistribution models of KMP-type [19, 8]. The dualities and self-dualities of all these processes arise naturally from the symmetries which are built in the construction.

It is the aim of this paper to provide the asymmetric version of these models with (self)-duality property, via the study of the deformed quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$. This provides a new class of bulk-driven non-equilibrium systems with duality, which includes in particular an asymmetric version of the KMP model [19]. The diversity of models related to the classical $\mathscr{U}(\mathfrak{su}(1,1))$ will also appear here in the asymmetric context where we consider the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$.

1.2 Models and abbreviations

For the sake of simplicity, we will use the following acronyms in order to describe the class of new processes that arise from our construction.

(a) Discrete representations will provide interacting particle systems in the class of *Inclusion Processes.* For a parameter $k \in \mathbb{R}_+$, the Symmetric Inclusion Process version is denoted by SIP(k), and **ASIP**(q, k) is the corresponding asymmetric version, with asymmetry parameter $q \in (0, 1)$.

(b) Continuous representations give rise to diffusion processes in the class of *Brownian Energy Processes.* For $k \in \mathbb{R}_+$, the Symmetric Brownian Energy Process is denoted by

BEP(k), and **ABEP**(σ, k) is the asymmetric version with asymmetry parameter $\sigma > 0$.

(c) By instantaneous thermalization, redistribution models are obtained, where energy or particles are redistributed at Poisson event times. This class includes the thermalized version of ABEP(σ, k), which is denoted by Th-ABEP(σ, k). In the particular case $k = 1/2$ the Th-ABEP(σ, k) is called the asymmetric KMP (Kipnis-Marchioro-Presutti) model, denoted by **AKMP**(σ), which becomes the KMP model as $\sigma \to 0$. The instantaneous thermalization of the ASIP(q, k) yields the Th-ASIP(q, k) process.

1.3 Markov processes with algebraic structure

In [10] we constructed a generalization of the asymmetric exclusion process, allowing $2j$ particles per site with self-duality properties reminiscent of the self-duality of the standard ASEP found initially by Schütz in [26]. This construction followed a general scheme where one starts from the Casimir operator C of the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(2))$, and applies a coproduct to obtain an Hamiltonian $H_{i,i+1}$ working on the occupation number variables at sites i and $i+1$. The operator $H = \sum_{i=1}^{L} H_{i,i+1}$ then naturally allows a rich class of commuting operators (symmetries), obtained from the n-fold coproduct applied to any generator of the algebra. This operator H is not yet the generator of a Markov process. But H allows a strictly positive ground state, which can also be constructed from the symmetries applied to a trivial ground state. Via a ground state transformation, H can then be turned into a Markov generator L of a jump process where particles hop between nearest neighbor sites and at most $2j$ particles per site are allowed. The symmetries of H directly translate into the symmetries of L, which in turn directly translate into self-duality functions.

This construction is in principle applicable to every quantum Lie algebra with a non-trivial center. However, it is not guaranteed that a Markov generator can be obtained. This depends on the chosen representation of the generators of the algebra, and the choice of the co-product. Recently the construction has been applied to algebras with higher rank, such as $\mathscr{U}_q(\mathfrak{gl}(3))$ [3, 20] or $\mathscr{U}_q(\mathfrak{sp}(4))$ [20], yielding two-component asymmetric exclusion process with multiple conserved species of particles.

1.4 Informal description of main results

In [14] we introduced a class of processes with $\mathfrak{su}(1,1)$ symmetry which in fact arise from this construction for the Lie algebra $\mathscr{U}(\mathfrak{su}(1,1))$. In this paper we look for natural asymmetric versions of the processes constructed in [14], and [8]. In particular the natural asymmetric analogue of the KMP process is a target. The main results are the following

(a) *Self-duality of ASIP*(q, k). We proceed via the same construction as in [10] for the algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ to find the ASIP(q, k) which is the "correct" asymmetric analogue of the SIP(k). The parameter q tunes the asymmetry: $q \to 1$ gives back the SIP(k). This process is then via its construction self-dual with a non-local self-duality function.

(b) *Duality between ABEP*(σ, k) *and SIP*(k). We then show that in the limit $\epsilon \to 0$ where simultaneously the asymmetry is going to zero ($q = 1 - \epsilon\sigma$ tends to unity), and the number of particles to infinity $\eta_i = \lfloor \epsilon^{-1} x_i \rfloor$, we obtain a diffusion process ABEP(σ, k) which is reminiscent of the Wright-Fisher diffusion with mutation and a selective drift.

As a consequence of self-duality of ASIP(q,k) we show that this diffusion process is dual to the SIP(k), i.e., the dual process is symmetric, and the asymmetry is in the duality function. Notice that this is the first example of duality between a truly asymmetric system (i.e. bulk-driven) and a symmetric system (with zero current).

(c) *Duality of instantaneous thermalization models.* Finally, we then consider instantaneous thermalization of ABEP(σ,k) to obtain an asymmetric energy redistribution model of KMP type. Its dual is the instantaneous thermalization of the SIP(k) which for $k = 1/2$ is exactly the dual KMP process.

1.5 Organization of the paper

The rest of our paper is organized as follows. In section 2 we introduce the process ASIP(q,k). After discussing some limiting cases, we show that this process has reversible profile product measures on $\mathbb{Z}_+$ (but not on $\mathbb{Z}$).

In section 3 we consider the weak asymmetry limit of ASIP(q,k). This leads to the diffusion process ABEP(σ,k), that also has reversible inhomogeneous product measures on the half-line. We prove that ABEP(σ,k) is a genuine non-equilibrium asymmetric system in the sense that it has a non-zero average current. Nevertheless in the last part of section 3 we show that the ABEP(σ,k) can be mapped – via a global change of coordinates – to the BEP(k), which is a symmetric system with zero-current. In section 3.6 this is also explained in the framework of the representation theory of the classical Lie algebra $\mathscr{U}(\mathfrak{su}(1,1))$.

In section 4 we introduce the instantaneous thermalization limits of both ASIP(q,k) and ABEP(σ,j) which are a particle, resp. energy, redistribution model at Poisson event times. This provides asymmetric redistribution models of KMP type.

In section 5 we introduce the self-duality of the ASIP(q,k) and prove various other duality relations that follow from it. In particular, once the self-duality of ASIP(q,k) is obtained, duality of ABEP(σ,k) with SIP(k) follows from a limiting procedure which is proved in Section 5.2. In the limit of an infinite number of particles with weak-asymmetry, the original process scales to ABEP(σ,k), whereas in the dual process the asymmetry disappears because the number of particles is finite. Next the self-duality and duality of thermalized models is derived in Section 5.3.

In section 6 we illustrate the use of the duality relations in various computations of exponential moments of currents. Finally, the last section is devoted to the full construction of the ASIP(q,k) from a $\mathscr{U}_q(\mathfrak{su}(1,1))$ symmetric quantum Hamiltonian and the proof of self-duality from the symmetries of this Hamiltonian.

2 The Asymmetric Inclusion Process ASIP(q,k)

2.1 Basic notation

We will consider as underlying lattice the finite lattice $\Lambda_L = \{1,\ldots,L\}$ or the periodic lattice $\mathbb{T}_L = \mathbb{Z}/L\mathbb{Z}$. At the sites of Λ_L we allow an arbitrary number of particles. The particle system configuration space is $\Omega_L = \mathbb{N}^{\Lambda_L}$. Elements of Ω_L are denoted by η,ξ and for $\eta \in \Omega_L$, $i \in \Lambda_L$, we denote by $\eta_i \in \mathbb{N}$ the number of particles at site i. For $\eta \in \Omega_L$ and $i,j \in \Lambda_L$ such that $\eta_i > 0$, we denote by $\eta^{i,j}$ the configuration obtained from η by removing one particle from i and putting it at j.

We need some further notation of q-numbers. For $q \in (0,1)$ and $n \in \mathbb{N}_0$ we introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{2.1}$$

satisfying the property $\lim_{q\to 1}[n]_q = n$. The first q-number's are thus given by

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q + q^{-1}, \qquad [3]_q = q^2 + 1 + q^{-2}, \quad \ldots$$

We also introduce the q-factorial

$$[n]_q! := [n]_q \cdot [n-1]_q \cdot \cdots \cdot [1]_q \ ,$$

and the q-binomial coefficient

$$\binom{n}{m}_q := \frac{[n]_q!}{[m]_q![n-m]_q!} \ .$$

Further we denote

$$(a;q)_m := (1-a)(1-aq)\cdots(1-aq^{m-1}) \ . \tag{2.2}$$

2.2 The ASIP(q,k) process

We introduce the process in finite volume by specifying its generator.

Definition 2.1 (ASIP(q,k) process)**.**

1. *The ASIP(q,k) with closed boundary conditions is defined as the Markov process on Ω_L with generator defined on functions $f : \Omega_L \to \mathbb{R}$*

$$(\mathscr{L}^{ASIP(q,k)}_{(L)} f)(\eta) := \sum_{i=1}^{L-1} (\mathscr{L}^{ASIP(q,k)}_{i,i+1} f)(\eta) \qquad \textit{with}$$

$$\begin{aligned}(\mathscr{L}^{ASIP(q,k)}_{i,i+1} f)(\eta) \quad &:= \quad q^{\eta_i - \eta_{i+1} + (2k-1)} [\eta_i]_q [2k + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ \quad q^{\eta_i - \eta_{i+1} - (2k-1)} [2k + \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned} \tag{2.3}$$

2. *The ASIP(q,k) with periodic boundary conditions is defined as the Markov process on $\mathbb{N}^{\mathbb{T}_L}$ with generator*

$$(\mathscr{L}^{ASIP(q,k)}_{(\mathbb{T}_L)} f)(\eta) := \sum_{i \in \mathbb{T}_L} (\mathscr{L}^{ASIP(q,k)}_{i,i+1} f)(\eta) \tag{2.4}$$

Since in finite volume we always start with finitely many particles, and the total particle number is conserved, the process is automatically well defined as a finite state space continuous time Markov chain. Later on (see Section 6.1) we will consider expectations of the self-duality functions in the infinite volume limit. In this way we can deal with relevant infinite volume expectations without having to solve the full existence problem of the ASIP(q,k) in infinite volume for a generic initial data. This might actually be an hard problem due to the lack of monotonicity.

2.3 Limiting cases

The ASIP (q,k) degenerates to well known interacting particle systems when its parameters take the limiting values $q \to 1$ and $k \to \infty$ recovering the cases of symmetric evolution or totally asymmetric zero range interaction. Notice in particular that these two limits do not commute.

- Convergence to symmetric processes

  i) $\mathbf{q \to 1, k}$ **fixed:** The ASIP(q,k) reduces to the SIP(k), i.e. the Symmetric Inclusion Process with parameter k. All the results of the present paper apply also to this symmetric case. In particular, in the limit $q \to 1$, the self-duality functions that will be given in theorem 5.1 below converge to the self-duality functions of the SIP(k) (given in [8]).

  ii) $\mathbf{q \to 1, k \to \infty}$**:** Furthermore, when the symmetric inclusion process is time changed so that time is scaled down by a factor $1/2k$, then in the limit $k \to \infty$ the symmetric inclusion converges weakly in path space to a system of symmetric independent random walkers (moving at rate 1).

- Convergence to totally asymmetric processes

  iii) $\mathbf{k \to \infty, q}$ **fixed:** If the limit $k \to \infty$ is performed first, then a totally asymmetric system is obtained under proper time rescaling. Indeed, by multiplying the ASIP(q,k) generator by $(1-q^2)q^{4k-1}$ one has

$$
\begin{aligned}
(1-q^2)q^{4k-1}\left[\mathscr{L}^{ASIP}_{i,i+1} f\right](\eta) &= q^{4k}\,\frac{(q^{2\eta_i}-1)(q^{4k}-q^{-2\eta_{i+1}})}{(1-q^2)}\left[f(\eta^{i,i+1})-f(\eta)\right] \\
&+ \frac{(q^{-2\eta_{i+1}}-1)(1-q^{2\eta_i+4k})}{(q^{-2}-1)}\left[f(\eta^{i+1,i})-f(\eta)\right]
\end{aligned}
$$

  Therefore, considering the family of processes $y^{(k)}(t) := \{y^{(k)}_i(t)\}_{i\in\Lambda_L}$ labeled by $k \geq 0$ and defining

$$
y^{(k)}_i(t) := \eta_i((1-q^2)q^{4k-1}t)
$$

  one finds that in the limit $k \to \infty$ the process $y^{(k)}(t)$ converges weakly to the Totally Asymmetric Zero Range process $y(t)$ with generator given by:

$$
(\mathscr{L}^{q-\text{TAZRP}}_{su(1,1)} f)(y) = \sum_{i=1}^{L-1} \frac{q^{-2y_{i+1}}-1}{q^{-2}-1}\,[f(y^{i+1,i})-f(y)], \qquad f:\Omega_L \to \mathbb{R} \tag{2.5}
$$

  In this system, particles jump to the left only with rates that are monotone increasing functions of the occupation variable of the departure site. Note that the rates are unbounded for $y_{i+1} \to \infty$, nevertheless the process is well defined even in the infinite volume, as it belongs to the class considered in [2]. This is to be compared to the case of the deformed algebra $U_q(\mathfrak{sl}_2)$ [10] whose scaling limit with infinite spin is given by [4]

$$
(\mathscr{L}^{(q-\text{TAZRP})}_{su(2)} f)(y) = \sum_{i=1}^{L-1} \frac{1-q^{2y_i}}{1-q^2}\,[f(y^{i,i+1})-f(y)], \qquad f:\Omega_L \to \mathbb{R} \tag{2.6}
$$

Here particles jump to the right only with rates that are also a monotonous increasing function of the occupation variable of the departure site, however now it is a bounded function approaching 1 in the limit $y_i \to \infty$. In [12] it is proved that the totally asymmetric zero range process (2.6) is in the KPZ universality class. It is an interesting open problem to prove or disprove that the same conclusion holds true for (2.5) [23]. We remark that the rates of (2.5) are (discrete) convex function and this also translates into convexity of the stationary current $j(\rho)$ as a function of the density ρ, whereas for (2.6) we have concave relations.

iv) $\mathbf{k} \to \infty, \mathbf{q} \to \mathbf{1}$**:** In the limit $q \to 1$ the zero range process in (2.5) reduces to a system of totally asymmetric independent walkers. This is to be compared to item ii) where symmetric walkers were found if the two limits were performed in the reversed order.

2.4 Reversible profile product measures

Here we describe the reversible measures of ASIP(q, k).

Theorem 2.1 (Reversible measures of ASIP(q, k))**.** *For all $L \in \mathbb{N}, L \geq 2$, the following results hold true:*

1.) the ASIP(q, k) on Λ_L with closed boundary conditions admits a family labeled by α of reversible product measures with marginals given by

$$\mathbb{P}^{(\alpha)}(\eta_i = n) = \frac{\alpha^n}{Z_i^{(\alpha)}} \binom{n+2k-1}{n}_q \cdot q^{4kin} \qquad n \in \mathbb{N} \tag{2.7}$$

for $i \in \Lambda_L$ and $\alpha \in [0, q^{-(2k+1)})$ (with the convention $\binom{2k-1}{0}_q = 1$). The normalization is

$$Z_i^{(\alpha)} = \sum_{n=0}^{+\infty} \binom{n+2k-1}{n}_q \cdot \alpha^n q^{4kin} = \frac{1}{(\alpha q^{4ki-(2k-1)}; q^2)_{2k}} \tag{2.8}$$

and for this measure

$$\mathbb{E}^{(\alpha)}(\eta_i) = \sum_{l=0}^{2k-1} \frac{1}{q^{-2l}(\alpha q^{4ki-2k+1})^{-1} - 1}\,. \tag{2.9}$$

2.) The ASIP(q, k) process on the torus $\mathbb{T}_L$ with periodic boundary condition does not admit homogeneous product measures.

Proof. The proof of item 2.) is similar to the proof of Theorem 3.1, item d) in [10] and we refer the reader to that paper for all details. To prove item 1.) consider the detailed balance relation

$$\mu(\eta) c_q(\eta, \eta^{i,i+1}) = \mu(\eta^{i,i+1}) c_q(\eta^{i,i+1}, \eta) \tag{2.10}$$

where the hopping rates are given by

$$c_q(\eta, \eta^{i,i+1}) = q^{\eta_i - \eta_{i+1} + 2k-1} [\eta_i]_q [2k + \eta_{i+1}]_q$$

$$c_q(\eta^{i,i+1},\eta) = q^{\eta_i-\eta_{i+1}-2k-1}[2k+\eta_i-1]_q[\eta_{i+1}+1]_q$$

and μ denotes a reversible measure. Suppose now that μ is a product measure of the form $\mu = \otimes_{i=1}^L \mu_i$. Then (2.10) holds if and only if

$$\mu_i(\eta_i-1)\mu_{i+1}(\eta_{i+1}+1)q^{-2k}[2k+\eta_i-1]_q[\eta_{i+1}+1]_q = \mu_i(\eta_i)\mu_{i+1}(\eta_{i+1})q^{2k}[\eta_i]_q[2k+\eta_{i+1}]_q \quad (2.11)$$

which implies that there exists $\alpha \in \mathbb{R}$ so that for all $i \in \Lambda_L$

$$\frac{\mu_i(n)}{\mu_i(n-1)} = \alpha q^{4ki}\frac{[2k+n-1]_q}{[n]_q}\ . \quad (2.12)$$

Then (2.7) follows from (2.12) after using an induction argument on n. The normalization $Z_i^{(\alpha)}$ is computed by using Corollary 10.2.2 of [1]. We have that

$$Z_i^{(\alpha)} < \infty \quad \text{if and only if} \quad 0 \le \alpha < q^{-4ki+(2k-1)} \quad \text{for any } i \in \Lambda_L \quad (2.13)$$

As a consequence (since $q < 1$ and $i = 1$ is the worst case) α must belong to the interval $[0, q^{-(2k+1)})$. The expectation (2.9) is obtained by exploiting the identity

$$\mathbb{E}^{(\alpha)}(\eta_i) = \alpha\frac{d}{d\alpha}\log Z_i^{(\alpha)}.$$

□

The following comments are in order:

i) *vanishing asymmetry*: in the limit $q \to 1$ the reversible product measure of ASIP(q,k) converges to a product of Negative Binomial distributions with shape parameter $2k$ and success probability α, which are the reversible measures of the SIP(k) [8].

ii) *monotonicity of the profile:* the average occupation number $\mathbb{E}^{(\alpha)}(\eta_i)$ in formula (2.9) is a decreasing function of i, and $\lim_{i\to\infty}\mathbb{E}^{(\alpha)}(\eta_i) = 0$.

iii) *infinite volume*: the reversible product measures with marginal (2.7) are also well-defined in the limit $L \to \infty$. One could go further to $[-M,\infty) \cap \mathbb{Z}$ for $\alpha < q^{4kM+2k-1}$ (but not to the full line $\mathbb{Z}$). These infinite volume measure concentrate on configurations with a finite number of particles, and thus are the analogue of the profile measures in the asymmetric exclusion process [21].

3 The Asymmetric Brownian Energy Process ABEP(σ,k)

Here we will take the limit of weak asymmetry $q = 1 - \epsilon\sigma \to 1$ ($\epsilon \to 0$) combined with the number of particles proportional to ϵ^{-1}, going to infinity, and work with rescaled particle numbers $x_i = \lfloor \epsilon\eta_i \rfloor$. Reminiscent of scaling limits in population dynamics, this leads to a diffusion process of Wright-Fisher type [9], with σ-dependent drift term, playing the role of a selective drift in the population dynamics language, or bulk driving term in the non-equilibrium statistical physics language.

3.1 Definition

We define the ABEP(q,k) process via its generator. It has state space $\mathscr{X}_L = (\mathbb{R}_+)^L$, $\mathbb{R}_+ := [0,+\infty)$. Configurations are denoted by $x \in \mathscr{X}_L$, with x_i being interpreted as the energy at site $i \in \Lambda_L$.

Definition 3.1 (ABEP(σ,k) process)**.**

1. *Let $\sigma > 0$ and $k \geq 0$. The Markov process ABEP(σ,k) on the state space $\mathscr{X}_L$ with closed boundary conditions is defined by the generator working on the core of smooth functions $f : \mathscr{X}_L \to \mathbb{R}$ via*

$$[\mathscr{L}_{(L)}^{ABEP(\sigma,k)} f](x) = \sum_{i=1}^{L-1} [\mathscr{L}_{i,i+1}^{ABEP(\sigma,k)} f](x) \tag{3.1}$$

with

$$\left[\mathscr{L}_{i,i+1}^{ABEP(\sigma,k)} f\right](x) = \frac{1}{4\sigma^2}(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 f(x)$$
$$-\frac{1}{2\sigma}\left\{(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1) + 2k\left(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}\right)\right\}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) f(x)$$

2. *The ABEP(σ,k) with periodic boundary conditions is defined as the Markov process on $\mathbb{R}_+^{\mathbb{T}_L}$ with generator*

$$[\mathscr{L}_{(\mathbb{T}_L)}^{ABEP(\sigma,k)} f](x) := \sum_{i \in \mathbb{T}_L} [\mathscr{L}_{i,i+1}^{ABEP(\sigma,k)} f](x) \tag{3.2}$$

The ABEP(σ,k) is a genuine asymmetric non-equilibrium system, in the sense that its translation-invariant stationary state may sustain a non-zero current. To see this, let $\mathbb{E}$ denote expectation with respect to the translation invariant measure for the ABEP(σ,k) on $\mathbb{T}_L$. Let $f_i(x) := x_i$, then from (3.2) we have

$$[\mathscr{L}^{ABEP(\sigma,k)} f_i](x) = \Theta_{i,i+1}(x) - \Theta_{i-1,i}(x) \tag{3.3}$$

with

$$\Theta_{i,i+1}(x) = -\frac{1}{2\sigma}\left\{(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1) + 2k\left(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}\right)\right\} \tag{3.4}$$

So we have

$$\frac{d}{dt}\mathbb{E}_x[f_i(x(t))] = \mathbb{E}_x[\Theta_{i,i+1}(x(t))] - \mathbb{E}_x[\Theta_{i-1,i}(x(t))]$$

and then, from the continuity equation we have that, in a translation invariant state, $\mathscr{J}_{i,i+1} := -\mathbb{E}[\Theta_{i,i+1}]$ is the instantaneous stationary current over the edge $(i,i+1)$. Thus we have the following

PROPOSITION 3.1 (Non-zero current of ABEP(σ,k))**.**

$$\mathscr{J}_{i,i+1} = -\mathbb{E}\left[\Theta_{i,i+1}\right] < 0 \qquad \textit{if} \quad k > 1/2$$

and

$$\mathscr{J}_{i,i+1} = -\mathbb{E}\left[\Theta_{i,i+1}\right] > 0 \qquad \textit{if} \quad k = 0\ .$$

PROOF. In the case $k > 1/2$, taking expectation of (3.4) we obtain

$$\mathbb{E}\left[\Theta_{i,i+1}\right] = \frac{1}{2\sigma}\left\{(1-4k) + (2k-1)\mathbb{E}(e^{2\sigma x_{i+1}} + e^{-2\sigma x_i}) + \mathbb{E}(e^{2\sigma(x_{i+1}-x_i)})\right\}$$

Since expectation in the translation invariant stationary state of local variables are the same on each site and $\cosh(x) \geq 1$ one obtains

$$\mathbb{E}\left[\Theta_{i,+1}\right] \geq \frac{1}{2\sigma}\left\{(1-4k) + 2(2k-1) + \mathbb{E}\left[e^{2\sigma(x_{i+1}-x_i)}\right]\right\}$$

Furthermore, Jensen inequality and translation invariance implies that

$$\mathbb{E}\left[\Theta_{i,i+1}\right] > \frac{1}{2\sigma}\left\{(1-4k) + 2(2k-1) + 1\right\} = 0$$

In the case $k = 0$ one has

$$\mathbb{E}\left[\Theta_{i,i+1}\right] = \frac{1}{2\sigma}\,\mathbb{E}\Big[(1-e^{-2\sigma x_i})(1-e^{2\sigma x_{i+1}})\Big] < 0$$

which is negative because the function is negative a.s. $\square$

3.2 Limiting cases

- Symmetric processes

  i) $\sigma \to \mathbf{0}, \mathbf{k}$ **fixed:** we recover the Brownian Energy Process with parameter k, BEP(k) (see [8]) whose generator is

$$\mathscr{L}^{BEP(k)}_{i,i+1} = x_i x_{i+1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 - 2k(x_i - x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) \tag{3.5}$$

  ii) $\sigma \to \mathbf{0}, \mathbf{k} \to \infty$**:** under the time rescaling $t \to t/2k$, one finds that in the limit $k \to \infty$ the BEP(k) process scales to a symmetric deterministic system evolving with generator

$$\left[\mathscr{L}^{DEP}_{i,i+1} f\right](x) \;\;=\;\; -(x_i - x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) f(x) \tag{3.6}$$

  This deterministic system is symmetric in the sense that if the initial condition is given by $(x_i(0), x_{i+1}(0)) = (a, b)$ then the asymptotic solution is given by the fixed point $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ where the initial total energy $a + b$ is equally shared among the two sites.

- Wright-Fisher diffusion

  iii) $\sigma \simeq \mathbf{0}, \mathbf{k}$ **fixed:** the ABEP(σ, k) on the simplex can be read as a Wright Fisher model with mutation and selection, however we have not been able to find in the literature the specific form of selection appearing in (3.2) (see [9] for the analogous result when $\sigma = 0$). To first order in σ one recovers the standard Wright-Fisher model with constant mutation k and selection σ

$$\mathscr{L}^{WF(\sigma,k)} = x_i x_{i+1} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 - (2\sigma x_i x_{i+1} + 2k(x_i - x_{i+1})) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)$$

- Asymmetric Deterministic System

  iv) $\mathbf{k} \to \infty, \sigma$ **fixed:** if the limit $k \to \infty$ is taken directly on the ABEP(σ, k) then, by time rescaling $t \to t/2k$ one arrives at an asymmetric deterministic system with generator

$$\mathscr{L}_{i,i+1}^{ADEP^{(\sigma)}} = -\frac{1}{2\sigma} \left(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}\right) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right) \tag{3.7}$$

  This deterministic system is asymmetric in the sense that if the initial condition is given by $(x_i(0), x_{i+1}(0)) = (a, b)$ then the asymptotic solution is given by the fixed point

$$(A, B) := \left(\frac{1}{2\sigma} \ln \left(\frac{1 + e^{2\sigma(a+b)}}{2} \right), a + b - \frac{1}{2\sigma} \ln \left(\frac{1 + e^{2\sigma(a+b)}}{2} \right) \right)$$

  where $A > B$.

  v) $\mathbf{k} \to \infty, \sigma \to \mathbf{0}$**:** in the limit $\sigma \to 0$ (3.7) converges to (3.6) and one recovers again the symmetric equi-distribution between the two sites of DEP process with generator (3.6).

  vi) $\mathbf{k} \to \infty, \sigma \to \infty$**:** in the limit $\sigma \to \infty$ one has the totally asymmetric stationary solution $(a + b, 0)$.

3.3 The ABEP(σ, k) as a diffusion limit of ASIP(q, k).

Here we show that the ABEP(σ, k) arises from the ASIP(q, k) in a limit of vanishing asymmetry and infinite particle number.

Theorem 3.1 (Weak asymmetry limit of ASIP(q, k))**.** *Fix $T > 0$. Let $\{\eta^\epsilon(t) : 0 \leq T\}$ denote the ASIP$(1 - \sigma\epsilon, k)$ starting from initial condition $\eta^\epsilon(0)$. Assume that*

$$\lim_{\epsilon \to 0} \epsilon \eta^\epsilon(0) = x \in \mathscr{X}_L \tag{3.8}$$

Then as $\epsilon \to 0$, the process $\{\eta^\epsilon(t) : 0 \leq t \leq T\}$ converges weakly on path space to the ABEP(σ, k) starting from x.

PROOF. The proof follows the lines of the corresponding results in population dynamics literature, i.e., Taylor expansion of the generator and keeping the relevant orders. Indeed, by the Trotter-Kurtz theorem [21], we have to prove that on the core of the generator of the limiting process, we have convergence of generators. Because the generator is a sum of terms working on two variables, our theorem follows from the computational lemma below. □

LEMMA 3.1. *If $\eta^\epsilon \in \Omega_L$ is such that $\epsilon\eta^\epsilon \to x \in \mathscr{X}_L$ then, for every smooth function $F : \mathscr{X}_L \to \mathbb{R}$, and for every $i \in \{1,\dots,L-1\}$ we have*

$$\lim_{\epsilon\to 0}(\mathscr{L}^{ASIP(1-\epsilon\sigma,k)}_{i,i+1} F_\epsilon)(\eta^\epsilon) = \mathscr{L}^{ABEP(\sigma,k)}_{i,i+1} F(x) \tag{3.9}$$

where $F_\epsilon(\eta) = F(\epsilon\eta)$, $\eta \in \Omega_L$.

PROOF. Define $x^\epsilon = \epsilon\eta^\epsilon$. Then we have, by the regularity assumptions on F that

$$\begin{aligned}
& F_\epsilon((\eta^\epsilon)^{i,i+1}) - F_\epsilon(\eta) \\
= \;& \epsilon\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)F(x^\epsilon) + \epsilon^2\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 F(x^\epsilon) + O(\epsilon^3)
\end{aligned} \tag{3.10}$$

and similarly

$$\begin{aligned}
& F_\epsilon((\eta^\epsilon)^{i+1,i}) - F_\epsilon(\eta) \\
= \;& -\epsilon\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)F(x^\epsilon) + \epsilon^2\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 F(x^\epsilon) + O(\epsilon^3)
\end{aligned} \tag{3.11}$$

Then using $q = 1 - \epsilon\sigma$, and

$$(1-\epsilon\sigma)^{x_i^\epsilon/\epsilon} = e^{-\sigma x_i} - 2x_i\sigma^2 e^{-2\sigma x_i}\epsilon + O(\epsilon^2)$$

straightforward computations give

$$\left[\mathscr{L}^\epsilon_{i,i+1}F\right](x^\epsilon) \;=\; \left[B_\epsilon(x^\epsilon)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) + D_\epsilon(x^\epsilon)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2\right]F(x^\epsilon) + O(\epsilon)$$

with

$$\begin{aligned}
B_\epsilon(x) &= \frac{1}{2\sigma}\left\{(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1) + 2k\left(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}\right)\right\} + O(\epsilon) \\
D_\epsilon(x) &= \frac{1}{4\sigma^2}(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1) + O(\epsilon)
\end{aligned} \tag{3.12}$$

Then we recognize

$$\begin{aligned}
& \left[B_\epsilon(x^\epsilon)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) + D_\epsilon(x^\epsilon)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2\right]F(x^\epsilon) \\
= \;& \left(\mathscr{L}^{ABEP(\sigma,k)}_{i,i+1}F\right)(x^\epsilon)
\end{aligned}$$

which ends the proof of the lemma by the smoothness of F and because by assumption, $x^\epsilon \to x$. $\square$

The weak asymmetry limit can also be performed on the q-TAZRP. This yields a totally asymmetric deterministic system as described in the following theorem.

Theorem 3.2 (Weak asymmetry limit of q-TAZRP)**.** *Fix $T > 0$. Let $\{y^\epsilon(t) : 0 \le T\}$ denote the q^ϵ-TAZRP, $q_\epsilon := 1 - \sigma\epsilon$, with generator* (2.5) *and initial condition $y^\epsilon(0)$. Assume that*

$$\lim_{\epsilon\to 0} \epsilon y^\epsilon(0) = y \in \mathscr{X}_L \tag{3.13}$$

Then as $\epsilon \to 0$, the process $\{y^\epsilon(t) : 0 \le t \le T\}$ converges weakly on path space to the Totally Asymmetric Deterministic Energy Process, TADEP(σ) with generator

$$(\mathscr{L}_{i,i+1}^{TADEP} f)(z) = -\left(\frac{1-e^{2\sigma z_{i+1}}}{2\sigma}\right)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right) f(z), \qquad f : \mathbb{R}_+^L \to \mathbb{R} \tag{3.14}$$

initialized from the configuration y.

Proof. The proof is analogous to the proof of Theorem 3.1 $\square$

3.4 Reversible measure of the ABEP(σ, k)

Theorem 3.3 (ABEP(σ, k) reversible measures)**.** *For all $L \in \mathbb{N}, L \ge 2$, the ABEP($q, k$) on $\mathscr{X}_L$ with closed boundary conditions admits a family (labeled by $\gamma > -4\sigma k$) of reversible product measures with marginals given by*

$$\mu_i(x_i) := \frac{1}{\mathscr{Z}_i^{(\gamma)}} (1 - e^{-2\sigma x_i})^{(2k-1)} e^{-(4\sigma k i + \gamma) x_i} \qquad\qquad x_i \in \mathbb{R}^+ \tag{3.15}$$

for $i \in \Lambda_L$ and

$$\mathscr{Z}_i^{(\gamma)} = \frac{1}{2\sigma}\, Beta\left(2ki + \frac{\gamma}{2\sigma}, 2k\right) \tag{3.16}$$

Proof. The adjoint of the generator of the ABEP(σ, k) is given by

$$\left(\mathscr{L}_{(L)}^{ABEP^{(\sigma,k)}}\right)^* = \sum_{i=1}^{L-1} \left(\mathscr{L}_{i,i+1}^{ABEP}\right)^* \tag{3.17}$$

with

$$\begin{aligned}
\left(\mathscr{L}_{i,i+1}^{ABEP}\right)^* f &= \frac{1}{4\sigma^2}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 \left(\left(1-e^{-2\sigma x_i}\right)\left(e^{2\sigma x_{i+1}} - 1\right) f\right) \\
-\frac{1}{2\sigma}\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) &\left(\left\{\left(1-e^{-2\sigma x_i}\right)\left(e^{2\sigma x_{i+1}} - 1\right) + 2k\left[\left(1-e^{-2\sigma x_i}\right) - \left(e^{2\sigma x_{i+1}} - 1\right)\right]\right\} f\right)
\end{aligned}$$

Let μ be a product measure with $\mu(x) = \prod_{i=1}^L \mu_i(x_i)$, then in order for μ to be a stationary measure it is sufficient to impose that the conditions

$$\frac{1}{4\sigma^2}\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)\left(1-e^{-2\sigma x_i}\right)\left(e^{2\sigma x_{i+1}}-1\right)\mu(x)$$
$$-\frac{1}{2\sigma}\left\{\left(1-e^{-2\sigma x_i}\right)\left(e^{2\sigma x_{i+1}}-1\right)+2k\left[\left(1-e^{-2\sigma x_i}\right)-\left(e^{2\sigma x_{i+1}}-1\right)\right]\right\}\mu(x)=0$$

are satisfied for any $i \in \{1, \ldots, L-1\}$. This is true if and only if

$$\frac{\mu_i'(x_i)}{\mu_i(x_i)} - 2\sigma\,\frac{2k-e^{-2\sigma x_i}}{1-e^{-2\sigma x_i}} + \sigma = \frac{\mu_{i+1}'(x_{i+1})}{\mu_{i+1}(x_{i+1})} + 2\sigma\,\frac{e^{2\sigma x_{i+1}}-2k}{e^{2\sigma x_{i+1}}-1} - \sigma \tag{3.18}$$

for any $x_i, x_{i+1} \in \mathbb{R}^+$. The conditions (3.18) are verified if and only if the marginals $\mu_i(x)$ are of the form (3.15) for some $\gamma \in \mathbb{R}$, $\mathscr{Z}_i^{(\gamma)}$ is a normalization constant, and the constraint $\gamma > -4\sigma k$ is imposed in order to assure the integrability of $\mu(\cdot)$ on $\mathscr{X}_L$. Thus we have proved that the product measure with marginal (3.15) are stationary. One can also verify that for any $f : \mathscr{X}_L \to \mathbb{R}$

$$\mathscr{L}^{ABEP} f = \frac{1}{\mu}\left(\mathscr{L}^{ABEP}\right)^*(\mu f)$$

which then implies that the measure is reversible. $\square$

Remark 3.1. *In the limit $\sigma \to 0$ the reversible product measure of ABEP(σ, k) converges to a product of Gamma distributions with shape parameter $2k$ and scale parameter $1/\gamma$, which are the reversible homogeneous measures of the BEP(k) [8]. In the case $\sigma \neq 0$ the reversible product measure of ABEP(σ, k) has a decreasing average profile (see Proposition 4.1).*

3.5 Transforming the ABEP(σ, k) to BEP(k)

In this subsection we show that the ABEP(σ, k), which is an asymmetric process, can be mapped via a global change of coordinates to the BEP(k) process which is symmetric. Here we focus on the analytical aspects of such σ-dependent mapping. In Section 3.6 we will show that this map induces a conjugacy at the level of the underlying $\mathfrak{su}(1,1)$ algebra. This implies that the ABEP(q, k) generator has a classical (i.e. non deformed) $\mathfrak{su}(1,1)$ symmetry. This is a remarkable because ABEP(q, k) is a bulk-driven non-equilibrium process with non-zero average current (as it has been shown in Proposition 3.1) and yet is generator is an element of the classical $\mathfrak{su}(1,1)$ algebra.

Definition 3.2 (Partial energy). *We define the partial energy functions $E_i : \mathscr{X}_L \to \mathbb{R}_+$, $i \in \{1, \ldots, L+1\}$*

$$E_i(x) := \sum_{\ell=i}^{L} x_\ell, \qquad \textit{for } i \in \Lambda_L \qquad \textit{and} \qquad E_{L+1}(x) = 0. \tag{3.19}$$

We also define the total energy $E : \mathscr{X}_L \to \mathbb{R}_+$ as

$$E(x) := E_1(x).$$

Definition 3.3 (The mapping g). *We define the map* $g : \mathscr{X}_L \to \mathscr{X}_L$

$$g(x) := (g_i(x))_{i\in\Lambda_L} \qquad \textit{with} \quad g_i(x) := \frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \tag{3.20}$$

Notice that g does not have full range, i.e. $g[\mathscr{X}_L] \neq \mathscr{X}_L$. Indeed

$$E(g(x)) = \frac{1}{2\sigma}\left(1 - e^{-2\sigma E(x)}\right) \leq \frac{1}{2\sigma} \tag{3.21}$$

so that in particular $g[\mathscr{X}_L] \subseteq \{x \in \mathscr{X}_L : E(x) \leq 1/2\sigma\}$. Moreover g is a bijection from $\mathscr{X}_L$ to $g[\mathscr{X}_L]$. Indeed, for $z \in g[\mathscr{X}_L]$ we have

$$(g^{-1}(z))_i = \frac{1}{2\sigma}\ln\left\{\frac{1 - 2\sigma\sum_{j=i+1}^{L} z_j}{1 - 2\sigma\sum_{j=i}^{L} z_j}\right\} \tag{3.22}$$

Theorem 3.4 (Mapping from ABEP(σ, k) to BEP(k)). *Let $X(t) = (X_i(t))_{i\in\Lambda_L}$ be the ABEP(σ, k) process starting from $X(0) = x$, then the process $Z(t) := (Z_i(t))_{i\in\Lambda_L}$ defined by the change of variable $Z(t) := g(X(t))$ is the BEP(k) with initial condition $Z(0) = g(x)$.*

Proof. It is sufficient to prove that, for any $f : \mathscr{X}_L \to \mathbb{R}_+$ smooth, $x \in \mathscr{X}_L$ and g defined above

$$\left[\mathscr{L}^{\text{BEP}}_{i,i+1} f\right](g(x)) = [\mathscr{L}^{\text{ABEP}}_{i,i+1}(f \circ g)](x) \tag{3.23}$$

for any $i \in \Lambda_L$. Define $F := f \circ g$, then

$$\begin{aligned} & [\mathscr{L}^{\text{ABEP}}(f \circ g)](x) = [\mathscr{L}^{\text{ABEP}}(F)](x) = \\ = \; & \frac{1}{4\sigma^2}(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 F(x) \\ + \; & \frac{1}{2\sigma}\left\{(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + 2k\left(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}\right)\right\}\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) F(x) \end{aligned} \tag{3.24}$$

The computation of the Jacobian of g

$$\frac{\partial g_j}{\partial x_i}(x) = \begin{cases} -2\sigma g_j(x) & \text{for } j \leq i-1 \\ e^{-2\sigma E_j(x)} & \text{for } j = i \\ 0 & \text{for } j \geq i+1 \end{cases} \tag{3.25}$$

implies that

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) g_j(x) = \begin{cases} 0 & \text{for } j \leq i-1 \\ -e^{-2\sigma E_{i+1}(x)} & \text{for } j = i \\ e^{-2\sigma E_{i+1}(x)} & \text{for } j = i+1 \\ 0 & \text{for } j \geq i+2 \end{cases} \tag{3.26}$$

and

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) F(x) = e^{-2\sigma E_{i+1}(x)}\left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right) f\right](g(x)) \tag{3.27}$$

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 F(x) = -2\sigma e^{-2\sigma E_{i+1}(x)} \left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right) f\right](g(x)) + e^{-4\sigma E_{i+1}(x)} \left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right)^2 f\right](g(x)). \tag{3.28}$$

Then, using (3.27) and (3.28), (3.24) can be rewritten as

$$\begin{aligned}
&[\mathscr{L}_{i,i+1}^{\text{ABEP}}(f \circ g)](x) = \\
&= \frac{1}{4\sigma^2}(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1)e^{-4\sigma E_{i+1}(x)} \left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right)^2 f\right](g(x)) \\
&+ \left\{2\sigma + \frac{1}{2\sigma}\left((1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + 2k\left(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}\right)\right)\right\} e^{-2\sigma E_{i+1}(x)} \\
&\qquad \cdot \left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right) f\right](g(x))
\end{aligned}$$

Simplifying, this gives

$$\begin{aligned}
&[\mathscr{L}_{i,i+1}^{\text{ABEP}}(f \circ g)](x) = \\
&= \left\{\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \cdot \frac{e^{-2\sigma E_{i+2}(x)} - e^{-2\sigma E_{i+1}(x)}}{2\sigma} \left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right)^2 f\right](g(x)) \right. \\
&- \frac{k}{\sigma}\left(e^{-2\sigma E_i(x)} - 2e^{-2\sigma E_{i+1}(x)} + e^{-2\sigma E_{i+2}(x)}\right) \left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right) f\right](g(x)) \\
&= \left[\mathscr{L}_{i,i+1}^{\text{BEP}} f\right](g(x))
\end{aligned}$$

□

The ABEP(σ, k) has a single conservation law given by the total energy $E(x) = \sum_{i \in \Lambda_L} x_i$. As a consequence there exists an infinite family of invariant measures which is hereafter described.

PROPOSITION 3.2 (Microcanonical measure of ABEP(σ, k))**.** *The stationary measure of the ABEP(σ, k) process on Λ_L with given total energy E is unique and is given by the inhomogeneous product measure with marginals* (3.15) *conditioned to a total energy $E(x) = E$. More explicitly*

$$d\mu^{(E)}(y) = \frac{\prod_{i=1}^L \mu_i(y_i) \mathbf{1}_{\{\sum_{i \in \Lambda_L} y_i = E\}} dy_i}{\int \ldots \int \prod_{i=1}^L \mu_i(y_i) \mathbf{1}_{\{\sum_{i \in \Lambda_L} y_i = E\}} dy_i} \tag{3.29}$$

PROOF. We start by observing that the stationary measure of the BEP(k) process on Λ_L with given total energy $\mathscr{E}$ is unique and is given by a product of i.i.d. Gamma random variable $(X_i)_{i \in \Lambda_L}$ with shape parameter $2k$ conditioned to $\sum_{i \in \Lambda_l} X_i = \mathscr{E}$. This is a consequence of duality between BEP(k) and SIP(k) processes [14]. Furthermore, an explicit computation shows that the reversible measure of ABEP(σ, k) conditioned to energy E are transformed by the mapping g (see Definition 3.3) to the stationary measure of the BEP(k) with energy $\mathscr{E}$ given by

$$\mathscr{E} = \frac{1}{2\sigma}(1 - e^{-2\sigma E}) \ .$$

The uniqueness for ABEP(σ, k) follows from the uniqueness for BEP(σ, k) and the fact that g is a bijection from $\mathscr{X}_L$ to $g[\mathscr{X}_L]$. $\square$

3.6 The algebraic structure of ABEP(σ, k)

First we recall from [14] that the BEP(k) generator can be written in the form

$$\mathscr{L}^{BEP(k)} = \sum_{i=1}^{L-1} \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - K_i^o K_{i+1}^o + 2k^2 \right) \tag{3.30}$$

where

$$\begin{aligned} K_i^+ &= z_i \\ K_i^- &= z_i \frac{\partial^2}{\partial z_i^2} + 2k \frac{\partial}{\partial z_i} \\ K_i^o &= z_i \frac{\partial}{\partial z_i} + k \end{aligned} \tag{3.31}$$

is a representation of the classical $\mathfrak{su}(1,1)$ algebra. We show here that the ABEP(σ, k) has the same algebraic structure. This is proved by using a representation of $\mathfrak{su}(1,1)$ that is conjugated to (3.31) and is given by

$$\tilde{K}_i^a = C_g \circ K_i^a \circ C_{g^{-1}} \qquad \text{with} \quad a \in \{+, -, o\} \tag{3.32}$$

where g is the function of Definition 3.3 and

$$(C_{g^{-1}} f)(x) = (f \circ g^{-1})(x)$$

$$(C_g f)(x) = (f \circ g)(x) \ .$$

Explicitly one has

$$(\tilde{K}_i^a f)(x) = (K_i^a f \circ g^{-1})(g(x)) \qquad \text{with} \quad a \in \{+, -, o\} \tag{3.33}$$

Theorem 3.5 (Algebraic structure of ABEP(σ, k))**.** *The generator of the ABEP(σ, k) process is written as*

$$\mathscr{L}^{ABEP(\sigma,k)} = \sum_{i=1}^{L-1} \left(\tilde{K}_i^+ \tilde{K}_{i+1}^- + \tilde{K}_i^- \tilde{K}_{i+1}^+ - \tilde{K}_i^o \tilde{K}_{i+1}^o + 2k^2 \right) \tag{3.34}$$

where the operators $\tilde{K}_i^a$ with $a \in \{+, -, o\}$ are defined in (3.32) *and provide a representation of the* $\mathfrak{su}(1,1)$ *Lie algebra.*

Proof. The proof is a consequence of the following two results:

$$\mathscr{L}^{ABEP(\sigma,k)} = C_g \circ \mathscr{L}^{BEP(k)} \circ C_{g^{-1}} \tag{3.35}$$

and the operators $\tilde{K}_i^a$ with $a \in \{+,-,o\}$ satisfy the commutation relations of the $\mathfrak{su}(1,1)$ algebra. The first property is an immediate consequence of Theorem 3.4, as Eq. (3.35) is simply a rewriting of Eq. (3.23) by using the definition of C_g and $C_{g^{-1}}$. The second property can be obtained by the following elementary Lemma, which implies that the commutation relations of the $\tilde{K}_i^a$ operators with $a \in \{+,-,o\}$ are the same of the K_i^a operators with $a \in \{+,-,o\}$. □

Lemma 3.2. *Consider an operator A working on function $f : \mathscr{X}_L \to \mathbb{R}$ and let $g : \mathscr{X}_L \to X \subset \mathscr{X}_L$ be a bijection. Then defining*

$$\tilde{A} = C_g \circ A \circ C_{g^{-1}}$$

we have that $A \to \tilde{A}$ is an algebra homomorphism.

Proof. We need to verify that

$$\widetilde{A+B} = \tilde{A} + \tilde{B} \qquad \text{and} \qquad \widetilde{AB} = \tilde{A}\tilde{B}$$

The first is trivial, the second is proved as follows

$$\widetilde{AB} = C_g \circ AB \circ C_{g^{-1}} = \left(C_g \circ A \circ C_{g^{-1}}\right) \circ \left(C_g \circ B \circ C_{g^{-1}}\right) = \tilde{A}\tilde{B}$$

As a consequence

$$\widetilde{[A,B]} = [\tilde{A}, \tilde{B}] \ .$$

□

4 The Asymmetric KMP process, AKMP(σ)

4.1 Instantaneous Thermalizations

The procedure of instantaneous thermalization has been introduced in [14]. We consider a generator of the form

$$\mathscr{L} = \sum_i \mathscr{L}_{i,i+1} \tag{4.1}$$

where $\mathscr{L}_{i,i+1}$ is such that, for any initial condition (x_i, x_{i+1}), the corresponding process converges to a unique stationary distribution $\mu_{(x_i,x_{i+1})}$.

Definition 4.1 (Instantaneous thermalized process). *The instantaneous thermalization of the process with generator $\mathscr{L}$ in* (4.1) *is defined to be the process with generator*

$$\mathscr{A} = \sum_i \mathscr{A}_{i,i+1}$$

where

$$\begin{aligned}\mathscr{A}_{i,i+1}f &= \lim_{t\to\infty}(e^{t\mathscr{L}_{i,i+1}}f - f) \qquad (4.2)\\ &= \int [f(x_1,\dots,x_{i-1},y_i,y_{i+1},x_{i+2},\dots,x_L) - f(x_1,\dots,x_L)]d\mu_{(x_i,x_{i+1})}(y_i,y_{i+1})\end{aligned}$$

In words, in the process with generator $\mathscr{A}$ each edge $(i, i+1)$ is updated at rate one, and after update its variables are replaced by a sample of the stationary distribution of the process with generator $\mathscr{L}_{i,i+1}$ starting from (x_i, x_{i+1}). Notice that, by definition, if a measure is stationary for the process with generator $\mathscr{L}_{i,i+1}$ then it is also stationary for the process with generator $\mathscr{A}_{i,i+1}$.

An example of thermalized processes is the Th-BEP(k) process, where the local redistribution rule is

$$(x, y) \to (B(x+y), (1-B)(x+y)) \qquad (4.3)$$

with B a Beta$(2k, 2k)$ distributed random variable [9]. In particular for $k = 1/2$ this gives the KMP process [19] that has a uniform redistribution rule on $[0, 1]$. Among discrete models we mention the Th-SIP(k) process where the redistribution rule is

$$(n, m) \to (R, n+m-R) \qquad (4.4)$$

where R is Beta-Binomial$(n+m, 2k, 2k)$. For $k = 1/2$ this corresponds to discrete uniform distributions on $\{0, 1, \dots, n+m\}$. Other examples are described in [9]. In the following we introduce the asymmetric version of these redistribution models.

4.2 Thermalized Asymmetric Inclusion process Th-ASIP(q, k)

The instantaneous thermalization limit of the Asymmetric Inclusion process is obtained as follows. Imagine on each bond $(i, i+1)$ to run the ASIP(q, k) dynamics for an infinite amount of time. Then the total number of particles on the bond will be redistributed according to the stationary measure on that bond, conditioned to conservation of the total number of particles of the bond. We consider the independent random variables $(M_1, \dots, M_L)$ distributed according to the stationary measure of the ASIP(q, k) at equilibrium. Thus M_i and M_{i+1} are distributed according to

$$p_i^{(\alpha)}(\eta_i) := \mathbb{P}^{(\alpha)}(M_i = \eta_i) = \frac{\alpha^{\eta_i}}{Z_i^{(\alpha)}}\binom{\eta_i + 2k - 1}{\eta_i}_q \cdot q^{4ki\eta_i} \qquad \eta_i \in \mathbb{N} \qquad (4.5)$$

and

$$p_{i+1}^{(\alpha)}(\eta_{i+1}) := \mathbb{P}^{(\alpha)}(M_{i+1} = \eta_{i+1}) = \frac{\alpha^{\eta_{i+1}}}{Z_{i+1}^{(\alpha)}}\binom{\eta_{i+1} + 2k - 1}{\eta_{i+1}}_q \cdot q^{4k(i+1)\eta_{i+1}} \qquad \eta_{i+1} \in \mathbb{N} \qquad (4.6)$$

for some $\alpha \in [0, q^{-(2k+1)})$. Hence the distribution of M_i, given that the sum is fixed to $M_i + M_{i+1} = n+m$ has the following probability mass function:

$$\begin{aligned}
\nu_{q,k}^{ASIP}(r \,|\, n+m) &:= \mathbb{P}(M_i = r \mid M_i + M_{i+1} = n+m) \qquad (4.7)\\
&= \frac{p_i^{(\alpha)}(r) p_{i+1}^{(\alpha)}(n+m-r)}{\sum_{l=0}^{n+m} p_i^{(\alpha)}(l) p_{i+1}^{(\alpha)}(n+m-l)}\\
&= \widetilde{\mathscr{C}}_{q,k}(n+m)\, q^{-4kr} \binom{r+2k-1}{r}_q \cdot \binom{2k+n+m-r-1}{n+m-r}_q
\end{aligned}$$

where $r \in \mathbb{N}$ and $\widetilde{\mathscr{C}}_{q,k}(n+m)$ is a normalization constant.

Definition 4.2 (Th-ASIP(q,k) process)**.** *The Th-ASIP(q,k) process on Λ_L is defined as the thermalized discrete process with state space Ω_L and local redistribution rule*

$$(n,m) \to (R_q, n+m-R_q) \qquad (4.8)$$

where R_q has a q-deformed Beta-Binomial$(n+m, 2k, 2k)$ distribution with mass function (4.7). The generator of this process is given by

$$\begin{aligned}
&\mathscr{L}_{th}^{ASIP(q,k)} f(\eta)\\
&= \sum_{i=1}^{L-1} \sum_{r=0}^{\eta_i+\eta_{i+1}} \left[f(\eta_1, \dots, \eta_{i-1}, r, \eta_i + \eta_{i+1} - r, \eta_{i+2}, \dots, \eta_L) - f(\eta) \right] \, \nu_{q,k}^{ASIP}(r \,|\, \eta_i + \eta_{i+1})
\end{aligned} \qquad (4.9)$$

4.3 Thermalized Asymmetric Brownian energy process Th-ABEP(σ,k).

We define the instantaneous thermalization limit of the Asymmetric Brownian Energy process as follows. On each bond we run the ABEP(σ,k) for an infinite time. Then the energies on the bond will be redistributed according to the stationary measure on that bond, conditioned to the conservation of the total energy of the bond. If we take two independent random variables X_i and X_{i+1} with distributions as in (3.15), i.e.

$$\mu_i(x_i) := \frac{1}{\mathscr{Z}_i^{(\gamma)}} \, (1 - e^{-2\sigma x_i})^{(2k-1)} e^{-(4\sigma k i + \gamma) x_i} \qquad x_i \in \mathbb{R}^+ \qquad (4.10)$$

$$\mu_{i+1}(x_{i+1}) := \frac{1}{\mathscr{Z}_{i+1}^{(\gamma)}} \, (1 - e^{-2\sigma x_{i+1}})^{(2k-1)} e^{-(4\sigma k (i+1) + \gamma) x_{i+1}} \qquad x_{i+1} \in \mathbb{R}^+ \qquad (4.11)$$

then the distribution of X_i, given the sum fixed to $X_i + X_{i+1} = E$, has density

$$\begin{aligned}
p(x_i | X_i + X_{i+1} = E) &= \frac{\mu_i(x_i)\mu_{i+1}(E - x_i)}{\int_0^E \mu_i(x)\mu_{i+1}(E-x)\, dx}\\
&= \mathscr{C}_{\sigma,k}(E)\, e^{4\sigma k x_i} \left[\left(1 - e^{-2\sigma x_i}\right) \left(1 - e^{-2\sigma(E - x_i)}\right) \right]^{2k-1}
\end{aligned}$$

where $\mathscr{C}_{\sigma,k}(E)$ is a normalization constant. Equivalently, let $W_i := X_i/E$, then W_i is a random variable taking values on $[0,1]$. Conditioned to $X_i + X_{i+1} = E$, its density is given by

$$\nu_{\sigma,k}(w|E) = \widehat{\mathscr{C}}_{\sigma,k}(E)\, e^{2\sigma E w} \left\{ \left(e^{2\sigma E w} - 1\right) \left(1 - e^{-2\sigma E(1-w)}\right) \right\}^{2k-1} \tag{4.12}$$

with

$$\widehat{\mathscr{C}}_{\sigma,k}(E) := \int_0^1 e^{2\sigma E w} \left\{ \left(e^{2\sigma E w} - 1\right) \left(1 - e^{-2\sigma E(1-w)}\right) \right\}^{2k-1} dw \tag{4.13}$$

Definition 4.3 (Thermalized ABEP(σ,k))**.** *The Th-ABEP(σ,k) process on Λ_L is defined as the thermalized process with state space $\mathscr{X}_L$ and local redistribution rule*

$$(x,y) \to (B_\sigma(x+y), (1-B_\sigma)(x+y)) \tag{4.14}$$

where B_σ has a distribution with density function $\nu_{\sigma,k}(\cdot|x+y)$ in (4.12)*. Thus the generator of Th-ABEP(σ,k) is given by*

$$\mathscr{L}_{th}^{ABEP(\sigma,k)} f(x) = \tag{4.15}$$
$$= \sum_{i=1}^{L-1} \int_0^1 \left[f(x_1, \ldots, w(x_i + x_{i+1}), (1-w)(x_i + x_{i+1}), \ldots, x_L) - f(x)\right] \nu_{\sigma,k}(w|x_i + x_{i+1})\, dw$$

In the limit $\sigma \to 0$, the conditional density $\nu_{0^+,k}(\cdot|E)$ does not depend on E, and for any $E \geq 0$ we recover the Beta$(2k,2k)$ distribution with density

$$\nu_{0^+,k}(w|E) = \frac{1}{\text{Beta}(2k,2k)} \left[w(1-w)\right]^{2k-1} . \tag{4.16}$$

Then the generator $\mathscr{L}_{th}^{ABEP(0^+,k)}$ coincides with the generator of the thermalized Brownian Energy process Th-BEP(k) defined in equation (5.13) of [8].

The redistribution rule with the random variable B_σ in Definition 4.3 is truly asymmetric, meaning that - on average - the energy is moved to the left.

Proposition 4.1. *Let B_σ be the random variable on* $[0,1]$ *distributed with density* (4.12)*, then $\mathbb{E}[B_\sigma] \geq \frac{1}{2}$. As a consequence B_σ and $1 - B_\sigma$ are not equal in distribution and for $(X_1, \ldots, X_L)$ distributed according to the reversible product measure μ of ABEP(σ,k) defined in* (3.15)*, we have that the energy profile is decreasing, i.e.*

$$\mathbb{E}_\mu[X_i] \geq \mathbb{E}_\mu[X_{i+1}], \qquad \forall\, i \in \{1, \ldots, L-1\} . \tag{4.17}$$

Proof. Let $X = (X_1, X_2)$ be a two-dimensional random vector taking values in $\mathscr{X}_2$ distributed according to the microcanonical measure $\mu^{(E)}$ of ABEP(σ,k) with fixed total energy $E \geq 0$, defined in (3.29). Then, from Definition 4.3,

$$(X_1, X_2) \stackrel{d}{=} (EB_\sigma, E(1-B_\sigma)) \qquad \text{with} \qquad B_\sigma \sim \nu_{\sigma,k}(\cdot|E) \tag{4.18}$$

Then, as already remarked in the proof of Proposition 3.2, $Z := g(X)$ with $g(\cdot)$ as in Definition 3.3 is a two-dimensional random variable taking values in $g[\mathscr{X}_2] \subset \mathscr{X}_2$ and distributed according to the microcanonical measure of BEP(k) with fixed total energy $\mathscr{E} = \frac{1}{2\sigma}(1 - e^{-2\sigma E})$. It follows from (4.3) that

$$g(X) \stackrel{d}{=} (\mathscr{E}B, \mathscr{E}(1-B)) \qquad \text{with} \qquad B \sim \text{Beta}(2k, 2k) \; . \tag{4.19}$$

Then, by (3.22) we have

$$(1 - B_\sigma)E = (g^{-1}(Z))_2 = \frac{1}{2\sigma} \ln \left\{ \frac{1}{1 - 2\sigma(1-B)\mathscr{E}} \right\} \tag{4.20}$$

and therefore

$$B_\sigma = 1 + \frac{1}{2\sigma E} \ln \left(1 - B(1 - e^{-2\sigma E}) \right) \tag{4.21}$$

Put $2\sigma E = 1$ without loss of generality, for simplicity. Then to prove that $\mathbb{E}[B_\sigma] > 1/2$ we have to prove that

$$\mathbb{E}(1 + \ln(1 - B(1 - e^{-1}))) \geq \frac{1}{2}$$

Defining $a = 1 - e^{-1}$ we then have to prove that

$$\mathbb{E}(-\ln(1 - aB)) \leq \frac{1}{2} \tag{4.22}$$

It is useful to write

$$-\ln(1 - aB) = \sum_{n=1}^{\infty} \frac{a^n B^n}{n}$$

and remark that for a $Beta(\alpha, \alpha)$ distributed B one has

$$\mathbb{E}(B^n) = \prod_{r=0}^{n-1} \frac{\alpha + r}{2\alpha + r} \; .$$

So we have to prove that

$$\psi(\alpha, a) := \sum_{n=1}^{\infty} \frac{a^n}{n} \prod_{r=0}^{n-1} \frac{\alpha + r}{2\alpha + r} < 1/2$$

First consider the limit $\alpha \to \infty$ then we find

$$\lim_{\alpha \to \infty} \varphi(\alpha, a) = \sum_{n=1}^{\infty} \frac{a^n}{2^n n} = -\ln \left(1 - \frac{1}{2}(1 - e^{-1}) \right) = -\ln \left(\frac{1}{2} + \frac{e^{-1}}{2} \right) \approx 0.379 < 1/2$$

Next remark when $\alpha = 0$ the B is distributed like $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ which gives

$$\mathbb{E}(-\ln(1 - aB)) = -\frac{1}{2} \ln(e^{-1}) = \frac{1}{2}$$

Now we prove that ψ is monotonically decreasing in α. To see this notice that

$$\frac{d}{d\alpha}\frac{\alpha+r}{2\alpha+r} = \frac{-r}{(2\alpha+r)^2} < 0$$

So the derivative

$$\frac{d}{d\alpha}\psi(\alpha,a) = \sum_{n=1}^{\infty}\sum_{r'=0}^{n-1}\frac{a^n}{n}\left(\prod_{r=0,r\neq r'}^{n-1}\frac{\alpha+r}{2\alpha+r}\right)\frac{-r'}{(2\alpha+r)^2} < 0$$

Therefore $\psi(\alpha,a)$ is monotonically decreasing in α and $\psi(\alpha,a) \le \frac{1}{2}$. Thus the claim $\mathbb{E}[B_\sigma] > 1/2$ is proved.

Now let $X = (X_1, X_2)$ be a two-dimensional r.v. distributed according to the profile measure μ defined in (3.15) with $L = 2$ and with abuse of notation let $\nu_{\sigma,k}[B_\sigma|E] = \mathbb{E}[B_\sigma]$. Then we can write $X = (E\,B_\sigma, E(1-B_\sigma))$ where now E is a random variable. We have

$$\begin{aligned}\mathbb{E}_\mu[X_2] &= \mathbb{E}_\mu[\mathbb{E}_\mu[X_2|\,E]] = \mathbb{E}_\mu[\mathbb{E}_\mu[E(1-B_\sigma)|\,E]] = \mathbb{E}_\mu[E\,\nu_{\sigma,k}[(1-B_\sigma)|\,E]] \\ &\le \mathbb{E}_\mu[E\,\nu_{\sigma,k}[B_\sigma|\,E]] = \mathbb{E}_\mu[\mathbb{E}_\mu[X_1|\,E]] = \mathbb{E}_\mu[X_1] \end{aligned} \tag{4.23}$$

The proof can be easily generalized to the case $L \ge 2$, yielding (4.17). □

For $k = 1/2$ and $\sigma \to 0$ the Th-ABEP(σ,k) is exactly the KMP process [19]. For $k = 1/2$ and $\sigma > 0$

$$\nu_{\sigma,1/2}(w|E) = \frac{2\sigma E}{e^{2\sigma E}-1}e^{2\sigma E w}, \qquad w \in [0,1] \tag{4.24}$$

The Th-ABEP$(\sigma,\frac{1}{2})$ can therefore be considered as the natural asymmetric analogue of the KMP process. This justifies the following definition.

Definition 4.4 (AKMP(σ) process)**.** *We define the Asymmetric KMP with asymmetry parameter $\sigma \in \mathbb{R}_+$ on Λ_L as the process with generator given by:*

$$\mathscr{L}^{AKMP(\sigma)}f(x) = \sum_{i=1}^{L-1}\left\{\frac{2\sigma(x_i+x_{i+1})}{e^{2\sigma(x_i+x_{i+1})}-1}\cdot\right.$$
$$\left.\cdot\int_0^1 [f(x_1,\dots,w(x_i+x_{i+1}),(1-w)(x_i+x_{i+1}),\dots,x_L)-f(x)]\;e^{2\sigma w(x_i+x_{i+1})}\,dw\right\}$$

5 Duality relations

In this section we derive various duality properties of the processes introduced in the previous sections. We start by recalling the definition of *duality*.

Definition 5.1. *Let $\{X_t\}_{t\ge0}$, $\{\widehat{X}_t\}_{t\ge0}$ be two Markov processes with state spaces Ω and $\widehat{\Omega}$ and $D : \Omega\times\widehat{\Omega} \to \mathbb{R}$ a bounded measurable function. The processes $\{X_t\}_{t\ge0}$, $\{\widehat{X}_t\}_{t\ge0}$ are said to be dual with respect to D if*

$$\mathbb{E}_x\big[D(X_t,\widehat{x})\big] = \widehat{\mathbb{E}}_{\widehat{x}}\big[D(x,\widehat{X}_t)\big] \tag{5.1}$$

for all $x \in \Omega, \widehat{x} \in \hat{\Omega}$ and $t > 0$. In (5.1) $\mathbb{E}_x$ is the expectation with respect to the law of the $\{X_t\}_{t\geq 0}$ process started at x, while $\widehat{\mathbb{E}}_{\widehat{x}}$ denotes expectation with respect to the law of the $\{\widehat{X}_t\}_{t\geq 0}$ process initialized at $\widehat{x}$.

5.1 Self-duality of ASIP(q,k)

The basic duality relation is the self-duality of ASIP(q,k). This self-duality property is derived from a symmetry of the underlying Hamiltonian which is a sum of co-products of the Casimir operator. In [10] this construction was achieved for the algebra $\mathscr{U}_q(\mathfrak{su}(2))$, and from the Hamiltonian a Markov generator was constructed via a positive ground state. Here the construction and consequent symmetries is analogous, but for the algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$. For the proof of the following Theorem we refer to Section 7.3, where we implement the steps of [10] for the algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$.

Theorem 5.1 (Self-duality of the finite ASIP(q,k))**.** *The ASIP(q,k) on Λ_L with closed boundary conditions is self-dual with the following self-duality function*

$$D_{(L)}(\eta,\xi) = \prod_{i=1}^{L} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{\xi_i+2k-1}{\xi_i}_q} \cdot q^{(\eta_i-\xi_i)\left[2\sum_{m=1}^{i-1}\xi_m+\xi_i\right]-4ki\xi_i} \cdot \mathbf{1}_{\xi_i\leq\eta_i} \tag{5.2}$$

or, equivalently,

$$D_{(L)}(\eta,\xi) = \prod_{i=1}^{L} \frac{(q^{2(\eta_i-\xi_i+1)};q^2)_{\xi_i}}{(q^{4k};q^2)_{\xi_i}} \cdot q^{(\xi_i-4ki+2N_{i+1}(\eta))\xi_i} \cdot \mathbf{1}_{\xi_i\leq\eta_i} \tag{5.3}$$

with $(a;q)_m$ as defined in (2.2) *and*

$$N_i(\eta) := \sum_{k=i}^{L} \eta_k \,. \tag{5.4}$$

Remark 5.1. *For $n \in \mathbb{N}$, let $\xi^{(\ell_1,\ldots,\ell_n)}$ be the configurations with n particles located at sites $\ell_1,\ldots,\ell_n$. Then for the configuration $\xi^{(\ell)}$ with one particle at site ℓ*

$$D(\eta,\xi^{(\ell)}) = \frac{q^{-(4k\ell+1)}}{q^{2k}-q^{-2k}} \cdot (q^{2N_\ell(\eta)} - q^{2N_{\ell+1}(\eta)}) \tag{5.5}$$

and, more generally, for the configuration $\xi^{(\ell_1,\ldots,\ell_n)}$ with n particles at sites $\ell_1,\ldots,\ell_n$ with $\ell_i \neq \ell_j$

$$D(\eta,\xi^{(\ell_1,\ldots,\ell_n)}) = \frac{q^{-4k\sum_{m=1}^{n}\ell_m - n^2}}{(q^{2k}-q^{-2k})^n} \cdot \prod_{m=1}^{n} (q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_m+1}(\eta)})$$

The duality relation with duality function (5.3) makes sense in the limit $L \to \infty$. Indeed, if $N_i(\eta) = \infty$ for some i, then $\lim_{L\to\infty} D_{(L)}(\eta,\xi) = 0$ for all ξ with $\xi_i \neq 0$. If the initial configuration $\eta \in \Omega_\infty$ has a finite number of particles at the right of the origin, then from the duality relation, we deduce that it remains like this for all later times $t > 0$, which implies that $N_\ell(\eta_t) < \infty$ for all $t \geq 0$. Conversely, if η is such that $N_0(\eta) = \infty$, then $N_0(\eta_t) = \infty$ for all later times because, from the duality relation, $\mathbb{E}_\xi\left[D(\eta,\xi_t)\right] = 0$ for all $t > 0$. To extract some non-trivial informations from the duality relation in the infinite volume case, a suitable renormalization is required (see Section 6.1).

5.2 Duality between ABEP(σ, k) and SIP(k)

We remind the reader that in the limit of zero asymmetry $q \to 1$ the ASIP(q, k) converges to the SIP(k). Therefore from the self-duality of ASIP(q, k), and the fact that the ABEP(σ, k) arises as a limit of ASIP(q, k) with $q \to 1$, a duality between ABEP(σ, k) and SIP(k) follows.

Theorem 5.2 (Duality ABEP(σ, k) and SIP(k))**.** *The ABEP(σ, k) on Λ_L with closed boundary conditions is dual to the SIP(k) on Λ_L with closed boundary conditions, with the following self-duality function*

$$D^{\sigma}_{(L)}(x, \xi) = \prod_{i \in \Lambda_L} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \right)^{\xi_i} \tag{5.6}$$

with $E_i(\cdot)$ the partial energy function defined in Definition 3.2.

Proof. The duality function in (5.6) is related to the duality function between BEP(k) and SIP(k), $D^0_{(L)}(x, \eta)$ (see e.g. Section 4.1 of [8]) by the following relation

$$D^{\sigma}_{(L)}(x, \xi) = D^0_{(L)}(g(x), \eta) \tag{5.7}$$

where $g(\cdot)$ is the map defined in (3.3). Thus, omitting the subscript (L) in the following, from (3.35) we have

$$\begin{aligned}
\left[\mathscr{L}^{\text{ABEP}(\sigma,k)} D^{\sigma}(\cdot, \eta)\right](x) &= \left[\mathscr{L}^{\text{ABEP}(\sigma,k)} \left(D^0(\cdot, \eta) \circ g\right)\right](x) \\
&= \left[\mathscr{L}^{\text{BEP}(k)} D^0(\cdot, \eta)\right](g(x)) \\
&= \left[\mathscr{L}^{\text{SIP}(k)} D^0(g(x), \cdot)\right](\eta) \\
&= \left[\mathscr{L}^{\text{SIP}(k)} D^{\sigma}(x, \cdot)\right](\eta)
\end{aligned} \tag{5.8}$$

and this proves the Theorem. □

Remark 5.2. *In the limit as $\sigma \to 0$ one recovers the duality $D^0_{(L)}(\cdot, \cdot)$ between BEP(k) and SIP(k). However it is remarkable here that for finite σ there is duality between a bulk driven asymmetric process, the ABEP(σ, k), and an equilibrium symmetric process, the SIP(k). Indeed, the asymmetry is hidden in the duality function. This is somewhat reminiscent of the dualities between systems with reservoirs and absorbing systems [8], where also the source of non-equilibrium, namely the different parameters of the reservoirs has been moved to the duality function.*

The following proposition explains how $D^{\sigma}_{(L)}(x, \xi)$ arises as the limit of ASIP(q, k) self-duality function for $q = 1 - N^{-1}\sigma$, $N \to \infty$.

Proposition 5.1. *For any fixed $L \geq 2$ we have*

$$\lim_{N \to \infty} \left(\frac{\sigma}{N}\right)^{|\xi|} D^{ASIP(1-\sigma/N, k)}_{(L)}(\lfloor Nx \rfloor, \xi) = D^{ABEP(\sigma,k)}_{(L)}(x, \xi) \tag{5.9}$$

where $D^{ASIP(q,k)}_{(L)}(\eta,\xi)$ *denotes the self-duality function of ASIP*(q,k) *defined in* (5.3) *and* $D^{ABEP(\sigma,k)}_{(L)}(x,\xi)$ *denotes the duality function defined in* (5.6).

Proof. Let

$$N := |\eta| := \sum_{i=1}^{L} \eta_i, \qquad q = 1 - \frac{\sigma}{N}, \qquad x := N^{-1}\eta, \tag{5.10}$$

then

$$D^{\mathrm{ASIP}(q,k)}_{(L)}(\eta,\xi) = \prod_{i=1}^{L} \frac{[\eta_i]_q[\eta_i-1]_q\dots[\eta_i-\xi_i+1]_q}{[2k+\xi_i-1]_q[2k+\xi_i-2]_q\dots[2k]_q} \cdot q^{(\eta_i-\xi_i)[2\sum_{m=1}^{i-1}\xi_m+\xi_i]-4ki\xi_i} \cdot \mathbf{1}_{\xi_i\le\eta_i} \tag{5.11}$$

Now, for any m

$$\begin{aligned}
[\eta_i - m]_{1-\frac{\sigma}{N}} &= [Nx_i - m]_{1-\frac{\sigma}{N}} \\
&= \frac{N}{2\sigma}\left[e^{\sigma x_i} - e^{-\sigma x_i} + O(N^{-1})\right] \\
&= \frac{N}{\sigma}\sinh(\sigma x_i) + O(1)
\end{aligned} \tag{5.12}$$

hence

$$\prod_{m=0}^{\xi_i-1} [Nx_i - m]_{1-\frac{\sigma}{N}} = \left(\frac{N}{\sigma}\,\sinh(\sigma x_i) + O(1)\right)^{\xi_i} \tag{5.13}$$

On the other hand

$$[2k+m]_{1-\frac{\sigma}{N}} = 2k+m+O(N^{-1}) \qquad \text{thus} \qquad \prod_{m=0}^{\xi_i-1}[2k+m]_{1-\frac{\sigma}{N}} = \frac{\Gamma(2k+\xi_i)}{\Gamma(2k)}+O(N^{-1}) \tag{5.14}$$

finally, let $f_i(\xi) := 2\sum_{m=1}^{i-1}\xi_m + \xi_i$ and $g_i(\xi) := -\xi_i\left[2\sum_{m=1}^{i-1}\xi_m + \xi_i\right] - 4ki\xi_i$ we have

$$q^{\eta_i f_i(\xi)} = \left(1-\frac{\sigma}{N}\right)^{Nx_i f_i(\xi)} = e^{-\sigma x_i f_i(\xi)} + O(N^{-1}), \quad \text{and} \quad q^{g(\xi)} = \left(1-\frac{\sigma}{N}\right)^{g(\xi)} = 1 + O(N^{-1}) \tag{5.15}$$

then (5.9) immediately follows. □

5.3 Duality for the instantaneous thermalizations

In this section we will prove that the self-duality of ASIP(q,k) and the duality between ABEP(σ,k) and SIP(k) imply duality properties also for the thermalized models.

Proposition 5.2. *If a process* $\{\eta(t) : t \ge 0\}$ *with generator* $\mathscr{L} = \sum_{i=1}^{L-1}\mathscr{L}_{i,i+1}$ *is dual to a process* $\{\xi(t) : t \ge 0\}$ *with generator* $\widehat{\mathscr{L}} = \sum_{i=1}^{L-1}\widehat{\mathscr{L}}_{i,i+1}$ *with duality function* $D(\cdot,\cdot)$ *in such a way that for all i*

$$\left[\mathscr{L}_{i,i+1}D(\cdot,\xi)\right](\eta) = [\widehat{\mathscr{L}}_{i,i+1}D(\eta,\cdot)](\xi)$$

then, if the instantaneous thermalization processes of η_t*, resp.* ξ_t *both exist, they are each other's dual with the same duality function* $D(\cdot,\cdot)$.

PROOF. Let $\mathscr{A}$, resp. $\widehat{\mathscr{A}}$ be the generators of the instantaneous thermalization of η_t, resp. ξ_t, then, from (4.2) we know that

$$\mathscr{A} = \sum_{i\in\Lambda_L} \mathscr{A}_{i,i+1}, \qquad \mathscr{A}_{i,i+1} = \lim_{t\to\infty}(e^{t\mathscr{L}_{i,i+1}} - I)$$

and

$$\widehat{\mathscr{A}} = \sum_{i\in\Lambda_L} \widehat{\mathscr{A}}_{i,i+1}, \qquad \widehat{\mathscr{A}}_{i,i+1} = \lim_{t\to\infty}(e^{t\widehat{\mathscr{L}}_{i,i+1}} - I)$$

where I denotes identity and where the exponential $e^{t\mathscr{L}_{i,i+1}}$ is the semigroup generated by $\mathscr{L}_{i,i+1}$ in the sense of the Hille Yosida theorem. Hence we immediately obtain that

$$\Big[(e^{t\mathscr{L}_{i,i+1}} - I)D(\cdot,\xi)\Big](\eta) = \Big[(e^{t\widehat{\mathscr{L}}_{i,i+1}} - I)D(\eta,\cdot)\Big](\xi)$$

which proves the result. $\square$

As a consequence of this Proposition we obtain duality between the thermalized ABEP(q,k) and the thermalized SIP(k) as well as self-duality of the thermalized ASIP(q,k).

THEOREM 5.3.

a) The Th-ASIP(q,k) with generator (4.9) *is self-dual with self-duality function given by* (5.2).

b) The Th-ABEP(σ,k) with generator (4.15) *is dual, with duality function* (5.6) *to the Th-SIP(k) in Λ_L whose generator is given by*

$$\mathscr{L}_{th}^{SIP(k)} f(\xi) = \tag{5.16}$$
$$= \sum_{i=1}^{L-1} \sum_{r=0}^{\xi_i+\xi_{i+1}} \left[f(\xi_1,\ldots,\xi_{i-1},r,\xi_i+\xi_{i+1}-r,\xi_{i+2},\ldots,\xi_L) - f(\xi)\right]\ \nu_k^{SIP}(r\,|\,\xi_i+\xi_{i+1})$$

where $\nu_k^{SIP}(r\,|\,n+m)$ is the probability density of a Beta-Binomial distribution of parameters $(n+m,2k,2k)$.

REMARK 5.3. *For $k=1/2$* (5.16) *gives the KMP-dual, i.e., the asymmetric KMP has the same dual as the symmetric KMP, but of course with different σ-dependent duality function given by*

$$D_{(L)}^{AKMP(\sigma)}(x,\xi) = \prod_{i\in\Lambda_L} \frac{1}{\xi_i!}\left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma}\right)^{\xi_i} \tag{5.17}$$

6 Applications to exponential moments of currents

The definition of the ASIP(q,k) process on the infinite lattice requires extra conditions on the initial data. Indeed, when the total number of particles is infinite, there is the possibility of the appearance of singularities, since a single site can accommodate an unbounded number of particles. By self-duality we can however make sense of expectations of duality functions in the infinite volume limit. This is the aim of the next section.

6.1 Infinite volume limit for ASIP(q,k)

In this section we approximate an infinite-volume configuration by a finite-volume configuration and we appropriately renormalize the self-duality function to avoid divergence in the thermodynamical limit.

Definition 6.1 (Good infinite-volume configuration).

a) We say that $\eta \in \mathbb{N}^{\mathbb{Z}}$ *is a* "good infinite-volume configuration" *for ASIP*(q,k) *iff for* $\eta^{(L)} \in \mathbb{N}^{\mathbb{Z}}$*,* $L \in \mathbb{N}$*, the restriction of* η *on* $[-L,L]$*, i.e.*

$$\eta_i^{(L)} = \begin{cases} \eta_i & \text{for} \quad i \in [-L,L] \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

the limit

$$\lim_{L\to\infty} \prod_{i\in\mathbb{Z}} q^{-2\xi_i N_{i+1}(\eta^{(L)})}\, \mathbb{E}_\xi \left[D(\eta^{(L)}, \xi(t)) \right] \tag{6.2}$$

exists and is finite for all $t \ge 0$ *and for any* $\xi \in \mathbb{N}^{\mathbb{Z}}$ *finite (i.e. such that* $\sum_{i\in\mathbb{Z}} \xi_i < \infty$*).*

b) Let μ *be a probability measure on* $\mathbb{N}^{\mathbb{Z}}$*, then we say that it is a* "good infinite-volume measure" *for ASIP*(q,k) *iff it concentrates on good infinite-volume configurations.*

Proposition 6.1.

1) If $\eta \in \mathbb{N}^{\mathbb{Z}}$ *is a "good infinite-volume configuration" for ASIP*(q,k) *and* $\xi^{(\ell_1,\dots,\ell_n)}$ *is the configurations with* n *particles located at sites* $\ell_1,\dots,\ell_n \in \mathbb{Z}$*, then the limit*

$$\lim_{L\to\infty} \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\, \mathbb{E}_{\eta^{(L)}} \left[D(\eta(t), \xi^{(\ell_1,\dots,\ell_n)}) \right] \tag{6.3}$$

is well-defined for all $t \ge 0$ *and is equal to*

$$\lim_{L\to\infty} \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\, \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[D(\eta^{(L)}, \xi(t)) \right] \tag{6.4}$$

2) If $\eta \in \mathbb{N}^{\mathbb{Z}}$ *is bounded, i.e.* $\sup_{i\in\mathbb{Z}} \eta_i < \infty$*, then it is a "good infinite-volume configuration".*

3) Let us denote by $\mathscr{N}_\lambda(t)$ *a Poisson process of rate* $\lambda > 0$*, and by* $\mathbf{E}[\cdot]$ *the expectation w.r. to its probability law. If* μ *is a probability measure on* $\mathbb{N}^{\mathbb{Z}}$ *such that for any* $\lambda > 0$ *the expectation*

$$\mathbb{E}_\mu \left[\mathbf{E} \left[e^{\sum_{i=1}^{\mathscr{N}_\lambda(t)} \eta_{\ell+i}} \right] \right] \tag{6.5}$$

is finite for all $t \ge 0$ *and for any* $\ell \in \mathbb{Z}$*, then* μ *is a "good infinite-volume measure".*

Proof.

1) If $\eta \in \mathbb{N}^{\mathbb{Z}}$ is a good infinite volume configuration, then the duality relation with duality function (5.3) makes sense after the following renormalization:

$$\mathbb{E}_{\eta^{(L)}}\left[D(\eta(t),\xi^{(\ell_1,\dots,\ell_n)})\right]\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})} = \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta^{(L)},\xi(t))\right]\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})} \tag{6.6}$$

then the first statement of the Theorem follows after taking the limit as $L \to \infty$ of (6.6).

2) Let ξ be a finite configuration in $\mathbb{N}^{\mathbb{Z}}$. We prove that for any bounded $\eta \in \mathbb{N}^{\mathbb{Z}}$ the family of functions

$$\mathscr{S}_L(t) := \prod_{i\in\mathbb{Z}} q^{-2\xi_i N_{i+1}(\eta^{(L)})}\,\mathbb{E}_{\xi}\left[D(\eta^{(L)},\xi(t))\right], \qquad L \in \mathbb{N} \tag{6.7}$$

is uniformly bounded. Without loss of generality we can suppose that $\xi = \xi^{(\ell_1,\dots,\ell_n)}$, for some $\{\ell_1,\dots,\ell_n\} \subset \mathbb{Z}$, $n \in \mathbb{N}$. Moreover we denote by $(\ell_1(t),\dots,\ell_n(t))$ the positions of the n ASIP(q,k) walkers starting at time $t = 0$ from $(\ell_1,\dots,\ell_n)$. We then have $\xi(t) = \xi^{(\ell_1(t),\dots,\ell_n(t))}$, and

$$\begin{aligned}\mathscr{S}_L(t) &= \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\,\mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta^{(L)},\xi(t))\right] = \\ &= \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[\prod_{i=1}^{L}\frac{(q^{2(\eta_i^{(L)}-\xi_i(t)+1)};q^2)_{\xi_i(t)}}{(q^{4k};q^2)_{\xi_i(t)}}\cdot q^{\xi_i^2(t)}\cdot \mathbf{1}_{\xi_i(t)\le\eta_i^{(L)}}\cdot\right.\\ &\left.\cdot\prod_{m=1}^{n} q^{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]}\right].\end{aligned}$$

As a consequence, since

$$(q^{2(\eta-\xi+1)};q^2)_\xi\cdot q^{\xi^2}\cdot\mathbf{1}_{\xi\le\eta}\le 1 \tag{6.8}$$

and

$$\sup_{\ell\le n}\frac{1}{(q^{4k};q^2)_\xi}\le c \tag{6.9}$$

for some $c > 0$, we have that there exists $C > 0$ such that

$$\left|\mathscr{S}_L(t)\right| \le C\,\mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[\prod_{m=1}^{n} q^{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]}\right] \tag{6.10}$$

for all $L \in \mathbb{N}$, $t \ge 0$. Then, from the Cauchy-Schwarz inequality, in order to find an upper bound for (6.10), it is sufficient to find an upper bound for

$$s_{L,m}(t) := \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[q^{\kappa\{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]\}}\right]$$

for any fixed $m \in \{1,\dots,n\}$ and $\kappa \in \mathbb{N}$. Now, let $M := \sup_{i\in\mathbb{Z}}\eta_i < \infty$, then

$$\left|N_{\ell_m(t)+1}(\eta^{(L)}) - N_{\ell_m+1}(\eta^{(L)})\right| \le M|\ell_m(t)-\ell_m|$$

hence there exists $C', \omega > 0$ such that

$$\left| s_{L,m}(t) \right| \le C' \, \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[e^{\omega |\ell_m(t) - \ell_m|} \right] \tag{6.11}$$

for any $L \in \mathbb{N}$, $t \ge 0$. Since $\xi(t)$ has a finite number of particles, for each $m \in \{1, \dots, n\}$ the process $|\ell_m(t) - \ell_m|$ is stochastically dominated by a Poisson process $\mathscr{N}(t)$ with parameter

$$\lambda := \max_{0 \le \eta, \eta' \le n} \{ q^{\eta - \eta' + (2k-1)} [\eta]_q [2k + \eta']_q \} \vee \max_{0 \le \eta, \eta' \le n} \{ q^{\eta - \eta' - (2k-1)} [2k + \eta]_q [\eta']_q \} \tag{6.12}$$

then the right hand side of (6.11) is less or equal than

$$\mathbf{E} \left[e^{\omega \mathscr{N}(t)} \right] = e^{-\lambda t} \sum_{i=0}^{\infty} e^{\omega i} \frac{(\lambda t)^i}{i!} < \infty \, . \tag{6.13}$$

This proves that $\mathscr{S}_L(t)$ is uniformly bounded.

3) Suppose that the probability measure μ satisfies (6.5). Then, in order to prove that it is a “good” measure, it is sufficient to show that

$$\lim_{L \to \infty} \mathbb{E}_{\mu} \left[\prod_{i \in \mathbb{Z}} q^{-2\xi_i N_{i+1}(\eta^{(L)})} \, \mathbb{E}_{\xi} \left[D(\eta^{(L)}, \xi(t)) \right] \right] < \infty \tag{6.14}$$

By exploiting the same arguments used in the proof of item 2), we claim that, in order to prove (6.14) it is sufficient to show that for each fixed $m = 1, \dots, n$, $\kappa > 0$, the function

$$\Theta_{L,m}(t) := \mathbb{E}_{\mu} \left[\mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[q^{\kappa \{ -4k\ell_m(t) + 2[N_{\ell_m(t)+1}(\eta^{(L)}) - N_{\ell_m+1}(\eta^{(L)})] \}} \right] \right] \tag{6.15}$$

is uniformly bounded. We have that

$$\begin{aligned} &\Theta_{L,m}(t) = \\ &= \mathbb{E}_{\mu} \left[\mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[q^{-4\kappa k \ell_m(t)} \left(q^{-2\kappa \sum_{i=\ell_m+1}^{\ell_m(t)} \eta_i^{(L)}} \mathbf{1}_{\ell_m < \ell_m(t)} + q^{2\kappa \sum_{i=\ell_m(t)+1}^{\ell_m} \eta_i^{(L)}} \mathbf{1}_{\ell_m(t) < \ell_m} \right) \right] \right] \\ &\le \mathbb{E}_{\mu} \left[\mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[q^{-4\kappa k \ell_m(t)} \left(q^{-2\kappa \sum_{i=1}^{\ell_m(t)-\ell_m} \eta_{i+\ell_m}^{(L)}} \mathbf{1}_{\ell_m < \ell_m(t)} + 1 \right) \right] \right] \, . \end{aligned}$$

Then the result follows as in proof of item 2) from the fact that the process $\ell_m(t) - \ell_m$ is stochastically dominated by a Poisson process of rate λ (6.12), and from the hypothesis (6.5).

□

Later on, if we write expectations in the infinite volume we always refer to the limiting procedure described above. Namely, for a “good infinite-volume configuration” $\eta \in \mathbb{N}^{\mathbb{Z}}$, with an abuse of notation we will write

$$\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta)} \, \mathbb{E}_{\eta} \left[D(\eta(t), \xi^{(\ell_1,\dots,\ell_n)}) \right] := \lim_{L \to \infty} \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})} \, \mathbb{E}_{\eta^{(L)}} \left[D(\eta(t), \xi^{(\ell_1,\dots,\ell_n)}) \right] \tag{6.16}$$

and

$$\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta)}\ \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta,\xi(t))\right] := \lim_{L\to\infty}\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\ \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta^{(L)},\xi(t))\right] \tag{6.17}$$

6.2 q-exponential moment of the current of ASIP(q,k)

We start by defining the current for the ASIP(q,k) process on $\mathbb{Z}$.

Definition 6.2 (Current)**.** *Let $\{\eta(t),\ t\geq 0\}$ be a càdlàg trajectory on the infinite-volume configuration space $\mathbb{N}^{\mathbb{Z}}$, then the total integrated current $J_i(t)$ in the time interval $[0,t]$ is defined as the net number of particles crossing the bond $(i-1,i)$ in the right direction. Namely, let $(t_i)_{i\in\mathbb{N}}$ be the sequence of the process jump times. Then*

$$J_i(t) = \sum_{k:t_k\in[0,t]}\left(\mathbf{1}_{\{\eta(t_k)=\eta(t_k^-)^{i-1,i}\}} - \mathbf{1}_{\{\eta(t_k)=\eta(t_k^-)^{i,i-1}\}}\right) \tag{6.18}$$

Lemma 6.1 (Current)**.** *The total integrated current of a càdlàg trajectory $(\eta(s))_{0\leq s\leq t}$ with $\eta(0)=\eta$ is given by*

$$J_i(t) = N_i(\eta(t)) - N_i(\eta) := \lim_{L\to\infty}\left(N_i(\eta^{(L)}(t)) - N_i(\eta^{(L)})\right) \tag{6.19}$$

where $N_i(\eta)$ is defined in (5.4) *and $\eta^{(L)}$ is defined in* (6.1)*. Moreover*

$$\lim_{i\to-\infty} J_i(t) = 0 \tag{6.20}$$

Proof. (6.19) immediately follows from the definition of $J_i(t)$, whereas (6.20) follows from the conservation of the total number of particles. □

Proposition 6.2 (Current q-exponential moment via a dual walker)**.** *Let $\eta\in\mathbb{N}^{\mathbb{Z}}$ a good infinite-volume configuration in the sense of Definition 6.1, then the first q-exponential moment of the current when the process is started from η at time $t=0$ is given by*

$$\mathbb{E}_\eta\left[q^{2J_i(t)}\right] = q^{2(N(\eta)-N_i(\eta))} - \sum_{n=-\infty}^{i-1} q^{4kn}\ \mathbf{E}_n\left[q^{-4km(t)}\left(1-q^{-2\eta_{m(t)}}\right)\ q^{2(N_{m(t)}(\eta)-N_i(\eta))}\right] \tag{6.21}$$

where $m(t)$ denotes a continuous time asymmetric random walker on $\mathbb{Z}$ jumping left at rate $q^{-2k}[2k]_q$ and jumping right at rate $q^{2k}[2k]_q$ and $\mathbf{E}_i$ denotes the expectation with respect to the law of $m(t)$ started at site $i\in\mathbb{Z}$ at time $t=0$. Furthermore $N(\eta)-N_i(\eta)=\sum_{n<i}\eta_n$ and the first term on the right hand side of (6.21) *is zero when there are infinitely many particles to the left of $i\in\mathbb{Z}$ in the configuration η.*

Proof. To prove (6.21) we consider the configuration $\xi^{(i)}\in\mathbb{N}^{\mathbb{Z}}$ with a single dual particle at site i. Since the ASIP(q,k) is self-dual the dynamics of the single dual particle is given an asymmetric random walk $m(t)$ on $\mathbb{Z}$ whose rates are computed from the process definition and coincides with those in the statement of the Proposition. From (6.16), (6.17) and item 1) of Proposition 6.1 we have that

$$q^{-2N_i(\eta)}\,\mathbb{E}_\eta\left[D(\eta(t),\xi^{(i)})\right] = \frac{q^{-(4ki+1)}}{q^{2k}-q^{-2k}}\,q^{-2N_i(\eta)}\,\mathbb{E}_\eta\left[q^{2N_i(\eta(t))}-q^{2N_{i+1}(\eta(t))}\right]$$

is equal to

$$q^{-2N_i(\eta)}\,\mathbb{E}_{\xi^{(i)}}\left[D(\eta,\xi^{(m(t))})\right] = q^{-2N_i(\eta)}\frac{q^{-1}}{q^{2k}-q^{-2k}}\,\mathbf{E}_i\left[q^{-4km(t)}(q^{2N_{m(t)}(\eta)}-q^{2N_{m(t)+1}(\eta)})\right]$$

Then from (6.19) we get

$$\begin{aligned}\mathbb{E}_\eta\left[q^{2J_i(t)}\right] &= q^{-2\eta_i}\,\mathbb{E}_\eta\left[q^{2J_{i+1}(t)}\right]\\ &+ q^{4ki}\,\mathbf{E}_i\left[q^{-4km(t)}(q^{2(N_{m(t)}(\eta)-N_i(\eta))}-q^{2(N_{m(t)+1}(\eta)-N_i(\eta))})\right]\end{aligned}\tag{6.22}$$

By iterating the relation in (6.22), for any $n\geq 0$ we get

$$\begin{aligned}&\mathbb{E}_\eta\left[q^{2J_{i+1}(t)}\right] = q^{2(N_{i-n}(\eta)-N_{i+1}(\eta))}\,\mathbb{E}_\eta\left[q^{2J_{i-n}(t)}\right]+\\ &-\sum_{j=0}^{n}q^{2(N_{i-j}(\eta)-N_{i+1}(\eta))}q^{4k(i-j)}\,\mathbf{E}_{i-j}\left[q^{-4km(t)}(q^{2(N_{m(t)}(\eta)-N_{i-j}(\eta))}-q^{2(N_{m(t)+1}(\eta)-N_{i-j}(\eta))})\right].\end{aligned}\tag{6.23}$$

By taking the limit $n\to\infty$ we get

$$\begin{aligned}\mathbb{E}_\eta\left[q^{2J_{i+1}(t)}\right] &= \lim_{n\to\infty}q^{2(N_{i-n}(\eta)-N_{i+1}(\eta))}\,\mathbb{E}_\eta\left[q^{2J_{i-n}(t)}\right]+\\ &-\sum_{j=0}^{\infty}q^{-2N_{i+1}(\eta)}q^{4k(i-j)}\,\mathbf{E}_{i-j}\left[q^{-4km(t)}(q^{2N_{m(t)}(\eta)}-q^{2N_{m(t)+1}(\eta)})\right]\end{aligned}$$

and using (6.20) we obtain (6.21). □

We continue with a lemma that is useful in the following.

Lemma 6.2. *Let $x(t)$ be the random walk on $\mathbb{Z}$ jumping to the right with rate $a\geq 0$ and to the left with rate $b\geq 0$, let $\alpha\in\mathbb{R}$, and $A\subseteq\mathbb{R}$ then*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbf{E}_0\left[\alpha^{x(t)}\,\middle|\,x(t)\in A\right] = \sup_{x\in A}\{x\log\alpha-\mathscr{I}(x)\}-\inf_{x\in A}\mathscr{I}(x)\tag{6.24}$$

with

$$\mathscr{I}(x) = (a+b)-\sqrt{x^2+4ab}+x\ln\left(\frac{x+\sqrt{x^2+4ab}}{2a}\right)\tag{6.25}$$

Proof. From large deviations theory [17] we know that $x(t)/t$, conditioned to $x(t)/t \in A$, satisfies a large deviation principle with rate function $\mathscr{I}(x) - \inf_{x\in A} \mathscr{I}(x)$ where $\mathscr{I}(x)$ is given by

$$\mathscr{I}(x) := \sup_z \{zx - \Lambda(z)\} \tag{6.26}$$

with

$$\Lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{zx(t)}\right] = a\left(e^z - 1\right) + b\left(e^{-z} - 1\right) \tag{6.27}$$

from which it easily follows (6.25). The application of Varadhan's lemma yields (6.24). □

Remark 6.1. *Let $m(t)$ be the random walk defined in Proposition 6.2, then* (6.24) *holds with*

$$\mathscr{I}(x) = [4k]_q - \sqrt{x^2 + (2[2k]_q)^2} + x \log\left\{\frac{1}{2[2k]_q q^{2k}}\left[x + \sqrt{x^2 + (2[2k]_q)^2}\right]\right\} \tag{6.28}$$

We denote by $\mathbb{E}^{\otimes\mu}$ the expectation of the ASIP(q,k) process on $\mathbb{Z}$ initialized with the homogeneous product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals μ at time 0, i.e.

$$\mathbb{E}^{\otimes\mu}[f(\eta(t))] = \sum_\eta \left(\otimes_{i\in\mathbb{Z}}\mu(\eta_i)\right) \mathbb{E}_\eta[f(\eta(t))] \, .$$

Proposition 6.3 (q-moment for product initial condition)**.** *Consider an homogeneous product probability measure μ on $\mathbb{N}$. Then, for the infinite volume ASIP(q,k), we have*

$$\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] = \mathbf{E}_0\left[\left(\frac{q^{-4k}}{\lambda_q}\right)^{m(t)} \mathbf{1}_{m(t)\le 0}\right] + \mathbf{E}_0\left[q^{-4km(t)}\left(\lambda_{1/q}^{m(t)} - \lambda_{1/q} + \lambda_q^{-1}\right) \mathbf{1}_{m(t)\ge 1}\right] \tag{6.29}$$

where $\lambda_y := \sum_{n=0}^\infty y^n \mu(n)$ and $m(t)$ is the random walk defined in Proposition 6.2. In particular we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\otimes\mu}[q^{2J_i(t)}] = \sup_{x\ge 0}\{x \log M_q - \mathscr{I}(x)\} - \inf_{x\ge 0} \mathscr{I}(x) \tag{6.30}$$

with $M_q := q^{-4k}\lambda_{1/q}$ and $\mathscr{I}(x)$ given by (6.28)*.*

Proof. It is easy to check that an homogeneous product measure μ verifies the condition (6.5) in Proposition 6.1, thus it is a good infinite-volume probability measure in the sense of Definition 6.1. For this reason we can apply Proposition 6.2, and from (6.21) we have

$$\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] = \int \otimes\mu(d\eta)\, \mathbb{E}_\eta\left[q^{2J_i(t)}\right]$$
$$= \int \otimes\mu(d\eta) q^{2(N(\eta)-N_i(\eta))} + \sum_{n=-\infty}^{i-1} q^{4kn} \int \otimes\mu(d\eta)\mathbf{E}_n\left[q^{-4km(t)}\left(q^{-2\eta_{m(t)}} - 1\right) q^{2(N_{m(t)}(\eta)-N_i(\eta))}\right]$$

Since

$$\int \otimes\mu(d\eta) q^{2(N_m(\eta)-N_i(\eta))} = \lambda_q^{i-m}\, \mathbf{1}_{\{m\le i\}} + \lambda_{1/q}^{m-i}\, \mathbf{1}_{\{m>i\}} \tag{6.31}$$

then, in particular, $\int \otimes\mu(d\eta) q^{2(N(\eta)-N_i(\eta))} = 0$ since $\lambda_q < 1$, where we recall the interpretation of $N(\eta) - N_i(\eta)$ from Proposition 6.2. Hence

$$\begin{aligned}
\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \sum_{n=-\infty}^{i-1} q^{4kn} \sum_{m\in\mathbb{Z}} \mathbf{P}_n\left(m(t)=m\right) q^{-4km} \int \otimes\mu(d\eta)\left[q^{2(N_{m+1}(\eta)-N_i(\eta))} - q^{2(N_m(\eta)-N_i(\eta))}\right] \\
&= \left(\lambda_q^{-1}-1\right) A(t) + \left(\lambda_{1/q}-1\right) B(t) \qquad (6.32)
\end{aligned}$$

with

$$A(t) := \sum_{n\le i-1} q^{4kn} \sum_{m\le i} \mathbf{P}_n\left(m(t)=m\right) q^{-4km} \lambda_q^{i-m} \qquad (6.33)$$

and

$$B(t) := \sum_{n\le i-1} q^{4kn} \sum_{m\ge i+1} \mathbf{P}_n\left(m(t)=m\right) q^{-4km} \lambda_{1/q}^{m-i} \qquad (6.34)$$

Now, let $\alpha := q^{-4k}\lambda_q^{-1}$, then

$$\begin{aligned}
A(t) &= \sum_{n\le i-1} q^{4kn}\lambda_q^i \sum_{m\le i} \mathbf{P}_n\left(m(t)=m\right)\alpha^m \\
&= \sum_{j\ge 1} \lambda_q^j \sum_{\bar m\le j} \mathbf{P}_0\left(m(t)=\bar m\right)\alpha^{\bar m} \\
&= \sum_{\bar m\le 0} \alpha^{\bar m}\mathbf{P}_0\left(m(t)=\bar m\right)\sum_{j\ge 1}\lambda_q^j + \sum_{\bar m\ge 1} \alpha^{\bar m}\mathbf{P}_0\left(m(t)=\bar m\right)\sum_{j\ge \bar m}\lambda_q^j \\
&= \frac{1}{1-\lambda_q}\left\{\lambda_q\,\mathbf{E}_0\left[\alpha^{m(t)}\,\mathbf{1}_{m(t)\le 0}\right] + \mathbf{E}_0\left[q^{-4km(t)}\,\mathbf{1}_{m(t)\ge 1}\right]\right\} \qquad (6.35)
\end{aligned}$$

Analogously one can prove that

$$B(t) = \frac{1}{\lambda_{1/q}-1}\left\{\mathbf{E}_0\left[\beta^{m(t)}\,\mathbf{1}_{m(t)\ge 2}\right] - \lambda_{1/q}\mathbf{E}_0\left[q^{-4km(t)}\,\mathbf{1}_{m(t)\ge 2}\right]\right\} \qquad (6.36)$$

with $\beta = q^{-4k}\lambda_{1/q}$ then (6.29) follows by combining (6.32), (6.35) and (6.36).

In order to prove (6.30) we use the fact that $m(t)$ has a Skellam distribution with parameters $([2k]_q q^{2k}t, [2k]_q q^{-2k}t)$, i.e. $m(t)$ is the difference of two independent Poisson random variables with those parameters. This implies that

$$\mathbf{E}_0\left[\left(\frac{q^{-4k}}{\lambda_q}\right)^{m(t)}\mathbf{1}_{m(t)\le 0}\right] = \mathbf{E}_0\left[\lambda_q^{m(t)}\mathbf{1}_{m(t)\ge 0}\right] .$$

Then we can rewrite (6.29) as

$$\begin{aligned}
\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \mathbf{E}_0\left[\lambda_q^{m(t)}\,\mathbf{1}_{m(t)\ge 1}\right] + \mathbf{P}_0\left(m(t)=0\right) \\
&+ \left(\lambda_q^{-1}-\lambda_{1/q}\right)\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\ge 1}\right] + \mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 1}\right] \\
&= \mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 0}\right]\left(1+\mathscr{E}_1(t)+\mathscr{E}_2(t)+\mathscr{E}_3(t)+\mathscr{E}_4(t)\right) \qquad (6.37)
\end{aligned}$$

with

$$\mathscr{E}_1(t) := \frac{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 0}\right]}, \qquad \mathscr{E}_2(t) := \frac{\mathbf{P}_0\left(m(t)=0\right)}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 0}\right]}$$

and

$$\mathscr{E}_3(t) := \frac{\mathbf{E}_0\left[\lambda_q^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 0}\right]}, \qquad \mathscr{E}_4(t) := \frac{\left(\lambda_q^{-1}-\lambda_{1/q}\right)\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 0}\right]} \tag{6.38}$$

To identify the leading term in (6.37) it remains to prove that, for each $i = 1, 2, 3$ there exists $c_i > 0$ such that

$$\sup_{t\geq 0}|\mathscr{E}_i(t)| \leq c_i \tag{6.39}$$

This would imply, making use of Lemma 6.2, the result in (6.30). The bound in (6.39) is immediate for $i = 1, 2, 3$. To prove it for $i = 4$ it is sufficient to show that there exists $c > 0$ such that

$$\lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\geq 1}\right] \leq c\,\mathbf{E}_0\left[\left(q^{-4k}\lambda_{1/q}\right)^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]. \tag{6.40}$$

This follows since there exists $m_* \geq 1$ such that for any $m \geq m_*$ $\lambda_q^{-1} \leq \lambda_{1/q}^m$ and then

$$\begin{aligned}
\lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\geq 1}\right] &\leq \lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{1\leq m(t)<m_*}\right] + \mathbf{E}_0\left[q^{-4km(t)}\lambda_{1/q}^{m(t)}\mathbf{1}_{m(t)\geq m_*}\right]\\
&\leq \lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{1\leq m(t)}\right] + \mathbf{E}_0\left[q^{-4km(t)}\lambda_{1/q}^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]\\
&\leq \left(1+\lambda_q^{-1}\right)\mathbf{E}_0\left[\left(q^{-4k}\lambda_{1/q}\right)^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]
\end{aligned} \tag{6.41}$$

This concludes the proof. □

6.3 Infinite volume limit for ABEP(σ, k)

Definition 6.3 (Good infinite-volume configuration)**.**

a) We say that $x \in \mathbb{R}_+^{\mathbb{Z}}$ is a "good infinite-volume configuration" *for ABEP(σ, k) iff for $x^{(L)} \in \mathbb{R}_+^{\mathbb{Z}}$, $L \in \mathbb{N}$, the restriction of x to $[-L, L]$, i.e.*

$$x_i^{(L)} = \begin{cases} x_i & \text{for} \quad i \in [-L, L] \\ 0 & \text{otherwise} \end{cases} \tag{6.42}$$

the limit

$$\lim_{L\to\infty}\prod_{i\in\mathbb{Z}} e^{2\sigma\xi_i E_{i+1}(x^{(L)})}\,\mathbb{E}_\xi\left[D^\sigma(x^{(L)}, \xi(t))\right] \tag{6.43}$$

exists and is finite for all $t \geq 0$ and for any $\xi \in \mathbb{N}^{\mathbb{Z}}$ finite (i.e. such that $\sum_{i\in\mathbb{Z}}\xi_i < \infty$).

b) Let μ be a probability measure on $\mathbb{R}_+^{\mathbb{Z}}$, then we say that it is a "good infinite-volume measure" *for ABEP(σ, k) iff it concentrates on good infinite-volume configurations.*

Proposition 6.4.

1) *If $x \in \mathbb{R}_+^{\mathbb{Z}}$ is a "good infinite-volume configuration" for ABEP(σ, k) and $\xi^{(\ell_1,\dots,\ell_n)}$ is the configurations with n particles located at sites $\ell_1, \dots, \ell_n \in \mathbb{Z}$, then the limit*

$$\lim_{L\to\infty} \prod_{m=1}^{n} e^{2\sigma E_{\ell_m+1}(x^{(L)})}\, \mathbb{E}_{x^{(L)}} \left[D^\sigma(x(t), \xi^{(\ell_1,\dots,\ell_n)})\right] \tag{6.44}$$

is well-defined for all $t \geq 0$ and is equal to

$$\lim_{L\to\infty} \prod_{m=1}^{n} e^{2\sigma E_{\ell_m+1}(x^{(L)})}\, \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[D^\sigma(x^{(L)}, \xi(t))\right] \tag{6.45}$$

2) *If $x \in \mathbb{R}_+^{\mathbb{Z}}$ is bounded, i.e. $\sup_{i\in\mathbb{Z}} x_i < \infty$, then it is a "good infinite-volume configuration" for ABEP(σ, k).*

3) *Let us denote by $\mathscr{N}_\lambda(t)$ a Poisson process of rate $\lambda > 0$, and by $\mathbf{E}[\cdot]$ the expectation w.r. to its probability law. If μ is a probability measure on $\mathbb{R}_+^{\mathbb{Z}}$ such that for any $\lambda > 0$ the expectation*

$$\mathbb{E}_\mu\left[\mathbf{E}\left[e^{\sum_{i=1}^{\mathscr{N}_\lambda(t)} x_{\ell+i}}\right]\right] \tag{6.46}$$

is finite for all $t \geq 0$ and for any $\ell \in \mathbb{Z}$, then μ is a "good infinite-volume measure" for ABEP(σ, k).

Proof. The proof is analogous to the proof of Proposition 6.1. □

Later on for a "good" infinite-volume configuration $x \in \mathbb{R}_+^{\mathbb{Z}}$ we will write

$$\prod_{i\in\mathbb{Z}} e^{2\sigma\xi_i E_{i+1}(x)}\, \mathbb{E}_\xi\left[D^\sigma(x, \xi(t))\right] := \lim_{L\to\infty} \prod_{i\in\mathbb{Z}} e^{2\sigma\xi_i E_{i+1}(x^{(L)})}\, \mathbb{E}_\xi\left[D^\sigma(x^{(L)}, \xi(t))\right] \tag{6.47}$$

and

$$\prod_{m=1}^{n} e^{2\sigma E_{\ell_m+1}(x)}\, \mathbb{E}_x\left[D^\sigma(x(t), \xi^{(\ell_1,\dots,\ell_n)})\right] := \lim_{L\to\infty} \prod_{m=1}^{n} e^{2\sigma E_{\ell_m+1}(x^{(L)})}\, \mathbb{E}_{x^{(L)}}\left[D^\sigma(x(t), \xi^{(\ell_1,\dots,\ell_n)})\right] \tag{6.48}$$

6.4 $e^{-\sigma}$-exponential moment of the current of ABEP(σ, k)

We start by defining the current for the ABEP(σ, k) process on $\mathbb{Z}$.

Definition 6.4 (Current). *Let $\{x(t),\ t \geq 0\}$ be a càdlàg trajectory on the infinite-volume configuration space $\mathbb{R}_+^{\mathbb{Z}}$, then the total integrated current $J_i(t)$ in the time interval $[0, t]$ is defined as total energy crossing the bond $(i-1, i)$ in the right direction.*

$$J_i(t) = E_i(x(t)) - E_i(x(0)) := \lim_{L\to\infty} \left(E_i(x^{(L)}(t)) - E_i(x^{(L)})\right) \tag{6.49}$$

where $E_i(x)$ is defined in (3.2) and $x^{(L)}$ as in (6.42).

Lemma 6.3 (Current)**.** *We have* $\lim_{i\to-\infty} J_i(t) = 0$.

Proof. It immediately follows from the conservation of the total energy. ☐

Proposition 6.5 (Current exponential moment via a dual walker)**.** *The first exponential moment of $J_i(t)$ when the process is started from a "good infinite-volume initial configuration" $x \in \mathbb{R}_+^{\mathbb{Z}}$ at time $t = 0$ is given by*

$$\mathbb{E}_x\left[e^{-2\sigma J_i(x(t))}\right] = e^{-4kt}\sum_{n\in\mathbb{Z}} e^{-2\sigma(E_n(x)-E_i(x))}\, I_{|n-i|}(4kt) \tag{6.50}$$

where $I_n(t)$ is the modified Bessel function.

Proof. Let $\xi^{(\ell)} \in \mathbb{R}_+^{\mathbb{Z}}$ be the configuration with a single particle at site ℓ. Since the ABEP(σ, k) is dual to the SIP$(2k)$ the dynamics of the single dual particle is given by a continuous time symmetric random walker $\ell(t)$ on $\mathbb{Z}$ jumping at rate $2k$. Since x is a good configuration we have that the normalized expectation

$$e^{2\sigma E_i(x)}\,\mathbb{E}_x\left[D(x(t),\xi^{(\ell)})\right] = \frac{1}{4k\sigma}\, e^{2\sigma E_i(x)}\,\mathbb{E}_x\left[e^{-2\sigma E_{\ell+1}(x(t))} - e^{-2\sigma E_\ell(x(t))}\right]$$

and, from the duality relation (5.5) this is also equal to:

$$e^{2\sigma E_i(x)}\,\mathbb{E}_{\xi^{(\ell)}}\left[D(x,\xi^{(\ell(t))})\right] = \frac{1}{4k\sigma}\, e^{2\sigma E_i(x)}\,\mathbf{E}_\ell\left[e^{-2\sigma E_{\ell(t)+1}(x)} - e^{-2\sigma E_{\ell(t)}(x)}\right]$$

where $\mathbf{E}_\ell$ denotes the expectation with respect to the law of $\ell(t)$ started at site $\ell \in \mathbb{Z}$ at time $t = 0$. As a consequence, for any $\ell \in \mathbb{Z}$

$$e^{2\sigma E_i(x)}\,\mathbb{E}_x\left[e^{-2\sigma E_{\ell+1}(x(t))}\right] = e^{2\sigma E_i(x)}\,\mathbb{E}_x\left[e^{-2\sigma E_\ell(x(t))}\right] + e^{2\sigma E_i(x)}\,\mathbf{E}_\ell\left[e^{-2\sigma E_{\ell(t)+1}(x)} - e^{-2\sigma E_{\ell(t)}(x)}\right] \tag{6.51}$$

from which it follows

$$\begin{aligned}
e^{2\sigma E_i(x)}\,\mathbb{E}_x\left[e^{-2\sigma E_i(x(t))}\right] &= e^{2\sigma E_i(x)}\sum_{\ell\le i-1}\mathbf{E}_\ell\left[e^{-2\sigma E_{\ell(t)+1}(x)} - e^{-2\sigma E_{\ell(t)}(x)}\right]\\
&= e^{2\sigma E_i(x)}\sum_{\ell\le i-1}\mathbf{E}_0\left[e^{-2\sigma E_{\ell(t)+\ell+1}(x)} - e^{-2\sigma E_{\ell(t)+\ell}(x)}\right]\\
&= e^{2\sigma E_i(x)}\sum_{m\le i}\mathbf{E}_0\left[e^{-2\sigma E_{\ell(t)+m}(x)}\right] - \sum_{\ell\le i-1}\mathbf{E}_0\left[e^{-2\sigma E_{\ell(t)+\ell}(x)}\right]\\
&= e^{2\sigma E_i(x)}\,\mathbf{E}_0\left[e^{-2\sigma E_{\ell(t)+i}(x)}\right]\\
&= e^{2\sigma E_i(x)}\,\mathbf{E}_i\left[e^{-2\sigma E_{\ell(t)}(x)}\right].
\end{aligned} \tag{6.52}$$

Thus we have arrived to

$$\mathbb{E}_x\left[e^{-2\sigma J_i(t)}\right] = \mathbf{E}_i\left[e^{-2\sigma\left(E_{\ell(t)}(x)-E_i(x)\right)}\right] \tag{6.53}$$

and the result (6.50) follows since

$$\mathbf{E}_i(f(\ell(t)) = \sum_{n\in\mathbb{Z}} f(n)\cdot \mathbf{P}_i(\ell(t)=n)$$

with

$$\begin{aligned}\mathbf{P}_i(\ell(t)=n) &= \mathbb{P}(\ell(t)=n \mid \ell(0)=i)\\ &= e^{-4kt} I_{|n-i|}(4kt)\end{aligned} \tag{6.54}$$

where $I_n(x)$ is the modified Bessel function. □

Remark 6.2. *Let $\ell(t)$ be a continuous time symmetric random walk on $\mathbb{Z}$ jumping at rate $2k$, then* (6.24) *holds with*

$$\mathscr{I}(x) = 4k - \sqrt{x^2+(4k)^2} + x\log\left\{\frac{1}{4k}\left[x+\sqrt{x^2+(4k)^2}\right]\right\} \tag{6.55}$$

We denote by $\mathbb{E}^{\otimes\mu}$ the expectation of the ABEP(σ,k) process on $\mathbb{Z}$ initialized with the omogeneous product measure on $\mathbb{R}^{\mathbb{Z}}$ with marginals μ at time 0, i.e.

$$\mathbb{E}^{\otimes\mu}[f(x(t))] = \int (\otimes_{i\in\mathbb{Z}}\mu(dx_i))\ \mathbb{E}_x[f(x(t))] \tag{6.56}$$

Proposition 6.6 (Exponential moment for product initial condition)**.** *Consider a probability measure μ on $\mathbb{R}^+$. Then, for the infinite volume ABEP(σ,k), we have*

$$\mathbb{E}^{\otimes\mu}\left[e^{-2\sigma J_i(t)}\right] = \mathbf{P}_0\left[\ell(t)=0\right] + \mathbf{E}_0\left[\left(\lambda_+^{\ell(t)}+\lambda_-^{\ell(t)}\right)\mathbf{1}_{\ell(t)\geq 1}\right] \tag{6.57}$$

where $\lambda_\pm := \int \mu(dy)e^{\pm 2\sigma y}$ and $\ell(t)$ is the random walk defined in Remark 6.2. In particular we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\otimes\mu}[e^{-2\sigma J_i(t)}] = \sup_{x\geq 0}\{x\log\lambda_+ - \mathscr{I}(x)\} - \inf_{x\geq 0}\mathscr{I}(x) \tag{6.58}$$

with $\mathscr{I}(x)$ given by (6.55)*.*

Proof. It is easy to check that an homogeneous product measure μ verifies the condition (6.46) in Proposition 6.1, thus it is a good infinite-volume probability measure for ABEP(σ,k) in the sense of Definition 6.3. Thus we can apply Proposition 6.5, in particular from (6.53) we have

$$\begin{aligned}&\mathbb{E}^{\otimes\mu}\left[e^{-2\sigma J_i(t)}\right] = \int \otimes\mu(dx)\,\mathbb{E}_x\left[e^{-2\sigma J_i(t)}\right]\\ &= \int\otimes\mu(dx)\mathbf{E}_i\left[e^{-2\sigma\left(E_{\ell(t)}(x)-E_i(x)\right)}\right] =\\ &= \sum_{n\in\mathbb{Z}}\mathbf{P}_i\left(\ell(t)=n\right)\ \int\otimes\mu(dx)e^{-2\sigma(E_n(x)-E_i(x))}\ .\end{aligned} \tag{6.59}$$

Since

$$\int \otimes\mu(d\eta)e^{-2\sigma(E_x(\eta)-E_i(\eta))} = \lambda_-^{i-n}\,\mathbf{1}_{\{n\le i\}} + \lambda_+^{n-i}\,\mathbf{1}_{\{n>i\}} \tag{6.60}$$

it follows that

$$\begin{aligned}
\mathbb{E}^{\otimes\mu}\left[e^{-2\sigma J_i(t)}\right] &= \sum_{n\le i}\mathbf{P}_i\left(\ell(t)=n\right)\lambda_-^{i-n} + \sum_{n\ge i+1}\mathbf{P}_i\left(\ell(t)=n\right)\lambda_+^{n-i} \\
&= \mathbf{E}_i\left[\lambda_-^{i-\ell(t)}\mathbf{1}_{\ell(t)\le i} + \lambda_+^{\ell(t)-i}\mathbf{1}_{\ell(t)\ge i+1}\right] \\
&= \mathbf{E}_0\left[\lambda_-^{-\ell(t)}\mathbf{1}_{\ell(t)\le 0} + \lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\ge 1}\right] \\
&= \mathbf{E}_0\left[\lambda_-^{\ell(t)}\mathbf{1}_{\ell(t)\ge 0} + \lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\ge 1}\right]
\end{aligned} \tag{6.61}$$

where the last identity follows from the symmetry of $\ell(t)$. Then (6.57) is proved.

In order to prove (6.58) we rewrite (6.57) as

$$\mathbb{E}^{\otimes\mu}\left[e^{-2\sigma J_i(t)}\right] = \mathbf{E}_0\left[\lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\ge 0}\right]\left(1+\mathscr{E}_1(t)+\mathscr{E}_2(t)\right) \tag{6.62}$$

with

$$\mathscr{E}_1(t) := \frac{\mathbf{E}_0\left[\left(\lambda_+^{\ell(t)}+\lambda_-^{\ell(t)}\right)\mathbf{1}_{\ell(t)\ge 1}\right]}{\mathbf{E}_0\left[\lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\ge 0}\right]}, \qquad \mathscr{E}_2(t) := \frac{\mathbf{P}_0\left(x(t)=0\right)}{\mathbf{E}_0\left[\lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\ge 0}\right]}$$

where for $i=1,2$ there exists $c_i>0$ such that

$$\sup_{t\ge 0}|\mathscr{E}_i(t)| \le c_i \tag{6.63}$$

This and the result of Remark 6.2 conclude the proof of(6.58). $\square$

7 Algebraic construction of ASIP(q,k) and proof of the self-duality

In this section we give the full proof of Theorem 5.1. It follows closely the lines of [10] however the algebra and co-product are different.

7.1 Algebraic structure and symmetries

The quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$

For $q\in(0,1)$ we consider the algebra with generators K^+, K^-, K^0 satisfying the commutation relations

$$[K^+,K^-] = -[2K^0]_q, \qquad [K^0,K^\pm] = \pm K^\pm\,, \tag{7.1}$$

where $[\cdot,\cdot]$ denotes the commutator, i.e. $[A,B]=AB-BA$, and

$$[2K^0]_q := \frac{q^{2K^0}-q^{-2K^0}}{q-q^{-1}} \; . \tag{7.2}$$

This is the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$, that in the limit $q\to 1$ reduces to the Lie algebra $\mathfrak{su}(1,1)$. The Casimir element is

$$C=[K^0]_q[K^0-1]_q-K^+K^- \tag{7.3}$$

A standard representation of the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ is given by

$$\begin{cases} K^+|n\rangle &= \sqrt{[\eta+2k]_q[\eta+1]_q}\;|n+1\rangle \\ K^-|n\rangle &= \sqrt{[\eta]_q[\eta+2k-1]_q}\;|n-1\rangle \\ K^0|n\rangle &= (\eta+k)\;|n\rangle \; . \end{cases} \tag{7.4}$$

$k\in\mathbb{N}$. Here the collection of column vectors $|n\rangle$, with $n\in\mathbb{N}$, denote the standard orthonormal basis with respect to the Euclidean scalar product, i.e. $|n\rangle=(0,\dots,0,1,0,\dots,0)^T$ with the element 1 in the n^{th} position and with the symbol ^{T} denoting transposition. Here and in the following, with abuse of notation, we use the same symbol for a linear operator and the matrix associated to it in a given basis. In the representation (7.4) the ladder operators K^+ and K^- are one the adjoint of the other, namely

$$(K^+)^*=K^- \tag{7.5}$$

and the Casimir element is given by the diagonal matrix

$$C|n\rangle=[k]_q[k-1]_q|n\rangle \; .$$

We also observe that the $\mathscr{U}_q(\mathfrak{su}(1,1))$ commutation relations in (7.1) can be rewritten as follows

$$\begin{aligned} q^{K_0}K^+ &= q\;K^+q^{K_0} \\ q^{K_0}K^- &= q^{-1}\;K^-q^{K_0} \\ [K^+,K^-] &= -[2K^0]_q \end{aligned} \tag{7.6}$$

Co-product structure

A co-product for the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ is defined as the map $\Delta:\mathscr{U}_q(\mathfrak{su}(1,1))\to\mathscr{U}_q(\mathfrak{su}(1,1))\otimes\mathscr{U}_q(\mathfrak{su}(1,1))$

$$\begin{aligned} \Delta(K^\pm) &= K^\pm\otimes q^{-K^0}+q^{K^0}\otimes K^\pm \; , \\ \Delta(K^0) &= K^0\otimes 1+1\otimes K^0 \; . \end{aligned} \tag{7.7}$$

The co-product is an isomorphism for the quantum Lie algebra $U_q(\mathfrak{sl}_2)$, i.e.

$$[\Delta(K^+),\Delta(K^-)]=-[2\Delta(K^0)]_q, \qquad [\Delta(K^0),\Delta(K^\pm)]=\pm\Delta(K^\pm) \; . \tag{7.8}$$

Moreover it can be easily checked that the co-product satisfies the co-associativity property

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \,. \tag{7.9}$$

Since we are interested in extended systems we will work with the tensor product over copies of the $\mathscr{U}_q(\mathfrak{su}(1,1))$ quantum algebra. We denote by K_i^+, K_i^-, K_i^0, with $i \in \mathbb{Z}$, the generators of the i^{th} copy. Obviously algebra elements of different copies commute. As a consequence of (7.9), one can define iteratively $\Delta^n : \mathscr{U}_q(\mathfrak{su}(1,1)) \to \mathscr{U}_q(\mathfrak{su}(1,1))^{\otimes(n+1)}$, i.e. higher power of Δ, as follows: for $n = 1$, from (7.7) we have

$$\begin{aligned} \Delta(K_i^\pm) &= K_i^\pm \otimes q^{-K_{i+1}^0} + q^{K_i^0} \otimes K_{i+1}^\pm \\ \Delta(K_i^0) &= K_i^0 \otimes 1 + 1 \otimes K_{i+1}^0 \,, \end{aligned} \tag{7.10}$$

for $n \geq 2$,

$$\begin{aligned} \Delta^n(K_i^\pm) &= \Delta^{n-1}(K_i^\pm) \otimes q^{-K_{n+i}^0} + q^{\Delta^{n-1}(K_i^0)} \otimes K_{n+i}^\pm \\ \Delta^n(K_i^0) &= \Delta^{n-1}(K_i^0) \otimes 1 + \underbrace{1 \otimes \ldots \otimes 1}_{n \text{ times}} \otimes K_{n+i}^0 \,. \end{aligned} \tag{7.11}$$

The quantum Hamiltonian

Starting from the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ and the co-product structure we would like to construct a linear operator (called "the quantum Hamiltonian" in the following and denoted by $H_{(L)}$ for a system of length L) with the following properties:

1. it is $\mathscr{U}_q(\mathfrak{su}(1,1))$ symmetric, i.e. it admits non-trivial symmetries constructed from the generators of the quantum algebra; the non-trivial symmetries can then be used to construct self-duality functions;
2. it can be associated to a continuos time Markov jump process, i.e. there exists a representation given by a matrix with non-negative out-of-diagonal elements (which can therefore be interpreted as the rates of an interacting particle systems) and with zero sum on each column.

A natural candidate for the quantum Hamiltonian operator is obtained by applying the co-product to the Casimir operator C in (7.3). Using the co-product definition (7.7), simple algebraic manipulations yield the following definition.

Definition 7.1 (Quantum Hamiltonian). *For every $L \in \mathbb{N}$, $L \geq 2$, we consider the operator $H_{(L)}$ defined by*

$$H_{(L)} := \sum_{i=1}^{L-1} H_{(L)}^{i,i+1} = \sum_{i=1}^{L-1} \left(h_{(L)}^{i,i+1} + c_{(L)} \right) \,, \tag{7.12}$$

where the two-site Hamiltonian is the sum of

$$c_{(L)} = \frac{(q^{2k} - q^{-2k})(q^{2k-1} - q^{-(2k-1)})}{(q - q^{-1})^2} \underbrace{1 \otimes \cdots \otimes 1}_{L \text{ times}} \tag{7.13}$$

and

$$h^{i,i+1}_{(L)} := \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\ times} \otimes \Delta(C_i) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i-1)\ times} \tag{7.14}$$

and, from (7.3) *and* (7.7),

$$\Delta(C_i) = \Delta(K_i^+)\Delta(K_i^-) - \Delta([K_i^0]_q)\Delta([K_i^0 - 1]_q) \tag{7.15}$$

Explicitely

$$\begin{aligned}\Delta(C_i) &= q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+\right\}q^{-K_{i+1}^0} + K_i^+K_i^- \otimes q^{-2K_{i+1}^0} + q^{2K_i^0} \otimes K_{i+1}^+K_{i+1}^- \\ &- \frac{1}{(q-q^{-1})^2}\left\{q^{-1}q^{2K_i^0} \otimes q^{2K_{i+1}^0} + qq^{-2K_i^0} \otimes q^{-2K_{i+1}^0} - (q+q^{-1})\right\} \end{aligned} \tag{7.16}$$

Remark 7.1. *By specializing* (7.16) *to the representation* (7.4) *we get*

$$\begin{aligned}\Delta(C_i) &= q^{K_i^0}\Bigg\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ + \\ &\quad - \frac{(q^k+q^{-k})(q^{k-1}+q^{-(k-1)})}{2(q-q^{-1})^2}\left(q^{K_i^0} - q^{-K_i^0}\right) \otimes \left(q^{K_{i+1}^0} - q^{-K_{i+1}^0}\right) \\ &\quad - \frac{(q^k-q^{-k})(q^{k-1}-q^{-(k-1)})}{2(q-q^{-1})^2}\left(q^{K_i^0} + q^{-K_i^0}\right) \otimes \left(q^{K_{i+1}^0} + q^{-K_{i+1}^0}\right)\Bigg\}q^{-K_{i+1}^0}\end{aligned} \tag{7.17}$$

Remark 7.2. *The diagonal operator* $c_{(L)}$ *in* (7.13) *has been added so that the ground state* $|0\rangle_{(L)} := \otimes_{i=1}^L |0\rangle_i$ *is a right eigenvector with eigenvalue zero, i.e.* $H_{(L)}|0\rangle_{(L)} = 0$ *as it is immediately seen using* (7.4).

Proposition 7.1. *In the representation* (7.4) *the operator* $H_{(L)}$ *is self-adjoint.*

Proof. It is enough to consider the non-diagonal part of $H_{(L)}$. Using (7.5) we have

$$\begin{aligned}&\left(q^{K_i^0}K_i^+ \otimes K_{i+1}^- q^{-K_{i+1}^0} + q^{K_i^0}K_i^- \otimes K_{i+1}^+ q^{-K_{i+1}^0}\right)^* \\ =\ & K_i^- q^{K_i^0} \otimes q^{-K_{i+1}^0}K_{i+1}^+ + K_i^+ q^{K_i^0} \otimes q^{-K_{i+1}^0}K_{i+1}^- \\ =\ & q^{K_i^0+1}K_i^- \otimes K_{i+1}^+ q^{-K_{i+1}^0-1} + q^{K_i^0-1}K_i^+ \otimes K_{i+1}^- q^{-K_{i+1}^0+1}\end{aligned}$$

where the last identity follows by using the commutation relations (7.6). This concludes the proof. □

Basic symmetries

It is easy to construct symmetries for the operator $H_{(L)}$ by using the property that the co-product is an isomorphism for the $\mathscr{U}_q(\mathfrak{su}(1,1))$ algebra.

Theorem 7.1 (Symmetries of $H_{(L)}$). *Recalling* (7.11), *we define the operators*

$$\begin{aligned} K^{\pm}_{(L)} &:= \Delta^{L-1}(K^{\pm}_1) = \sum_{i=1}^{L} q^{K^0_1}\otimes\cdots\otimes q^{K^0_{i-1}}\otimes K^{\pm}_i\otimes q^{-K^0_{i+1}}\otimes\ldots\otimes q^{-K^0_L}\ , \\ K^0_{(L)} &:= \Delta^{L-1}(K^0_1) = \sum_{i=1}^{L} \underbrace{1\otimes\cdots\otimes 1}_{(i-1)\ times}\otimes K^0_i\otimes\underbrace{1\otimes\cdots\otimes 1}_{(L-i)\ times}\ . \end{aligned} \tag{7.18}$$

They are symmetries of the Hamiltonian (7.12), *i.e.*

$$[H_{(L)},K^{\pm}_{(L)}] = [H_{(L)},K^0_{(L)}] = 0\ . \tag{7.19}$$

Proof. We proceed by induction and prove only the result for $K^{\pm}_{(L)}$ (the case $K^0_{(L)}$ is similar). By construction $K^{\pm}_{(2)} := \Delta(K^{\pm})$ are symmetries of the two-site Hamiltonian $H_{(2)}$. Indeed this is an immediate consequence of the fact that the co-product defined in (7.8) conserves the commutation relations and the Casimir operator (7.3) commutes with any other operator in the algebra :

$$[H_{(2)},K^{\pm}_{(2)}] = [\Delta(C_1),\Delta(K^{\pm}_1)] = \Delta([C_1,K^{\pm}_1]) = 0\ .$$

For the induction step assume now that it holds $[H_{(L-1)},K^{\pm}_{(L-1)}]=0$. We have

$$[H_{(L)},K^{\pm}_{(L)}] = [H_{(L-1)},K^{\pm}_{(L)}] + [h^{L-1,L}_{(L)},K^{\pm}_{(L)}] \tag{7.20}$$

The first term on the right hand side of (7.20) can be seen to be zero using (7.11) with $i=1$ and $n=L-1$:

$$[H_{(L-1)},K^{\pm}_{(L)}] = [H_{(L-1)},K^{\pm}_{(L-1)}q^{-K^0_L} + q^{K^0_{(L-1)}}K^{\pm}_L]$$

Distributing the commutator with the rule $[A,BC]=B[A,C]+[A,B]C$, the induction hypothesis and the fact that spins on different sites commute imply the claim. The second term on the right hand side of (7.20) is also seen to be zero by writing

$$[h^{L-1,L}_{(L)},K^{\pm}_{(L)}] = [h^{L-1,L}_{(L)},K^{\pm}_{(L-2)}q^{-\Delta(K^0_{L-1})} + q^{K^0_{(L-2)}}\Delta(K^{\pm}_{L-1})] = 0\ .$$

□

Remark 7.3. *In the case $q=1$, the quantum Hamiltonian in Definition 7.1 reduces to the (negative of the) well-known Heisenberg ferromagnetic quantum spin chain*

$$H_{(L)} = \sum_{i=1}^{L-1}\left(K^+_iK^-_{i+1} + K^-_iK^+_{i+1} - 2K^0_iK^0_{i+1} + 2k^2\right)\ , \tag{7.21}$$

with spins K_i satisfying the $\mathfrak{su}(1,1)$ Lie algebra. The symmetries of this Hamiltonian are given by

$$K^{\pm}_{(L)} = \sum_{i=1}^{L} K^{\pm}_i \qquad and \qquad K^{o}_{(L)} = \sum_{i=1}^{L} K^{o}_i\ .$$

7.2 Construction of ASIP(q, k) from the quantum Hamiltonian

In order to construct a Markov process from the quantum Hamiltonian $H_{(L)}$, we make use of the following Theorem which has been proven in [10].

Theorem 7.2 (Positive ground state transformation)**.** *Let A be a $|\Omega| \times |\Omega|$ matrix with non-negative off diagonal elements. Suppose there exists a column vector $e^{\psi} := g \in \mathbb{R}^{|\Omega|}$ with strictly positive entries and such that $Ag = 0$. Let us denote by G the diagonal matrix with entries $G(x, x) = g(x)$ for $x \in \Omega$. Then we have the following*

a) The matrix

$$\mathscr{L} = G^{-1}AG$$

with entries

$$\mathscr{L}(x, y) = \frac{A(x, y)g(y)}{g(x)}, \qquad x, y \in \Omega \times \Omega \tag{7.22}$$

is the generator of a Markov process $\{X_t : t \geq 0\}$ taking values on Ω.

b) S commutes with A if and only if $G^{-1}SG$ commutes with $\mathscr{L}$.

c) If $A = A^*$, where * denotes transposition, then the probability measure μ on Ω

$$\mu(x) = \frac{(g(x))^2}{\sum_{x\in\Omega}(g(x))^2} \tag{7.23}$$

is reversible for the process with generator $\mathscr{L}$.

Now we apply item a) of Theorem 7.2 with $A = H_{(L)}$. At this aim we need a non-trivial symmetry which yields a non-trivial ground state. Starting from the basic symmetries of $H_{(L)}$ described in Section 7.1, and inspired by the analysis of the symmetric case ($q = 1$), it will be convenient to consider the *exponential* of those symmetries.

The q-exponential and its pseudo-factorization

Definition 7.2 (q-exponential)**.** *We define the q-analog of the exponential function as*

$$\exp_q(x) := \sum_{n\geq 0} \frac{x^n}{\{n\}_q!} \tag{7.24}$$

where

$$\{n\}_q := \frac{1 - q^n}{1 - q} \tag{7.25}$$

Remark 7.4. *The q-numbers in (7.25) are related to the q-numbers in (2.1) by the relation $\{n\}_{q^2} = [n]_q q^{n-1}$. This implies $\{n\}_{q^2}! = [n]_q!\, q^{n(n-1)/2}$ and therefore*

$$\exp_{q^2}(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q!}\, q^{-n(n-1)/2} \tag{7.26}$$

Proposition 7.2 (Pseudo-factorization). *Let $\{g_1, \dots, g_L\}$ and $\{k_1, \dots, k_L\}$ be operators such that for $L \in \mathbb{N}$ and $g \in \mathbb{R}$*

$$k_i g_i = r g_i k_i \qquad \text{for} \quad i \in \Lambda_L \ . \tag{7.27}$$

Define

$$g^{(L)} := \sum_{i=1}^{L} k^{(i-1)} g_i, \quad \text{with} \quad k^{(i)} := k_1 \cdot \dots \cdot k_i \quad \text{for } i \geq 1 \text{ and} \quad k^{(0)} = 1, \tag{7.28}$$

then

$$\exp_r(g^{(L)}) = \exp_r(g_1) \cdot \exp_r(k^{(1)} g_2) \cdot \dots \cdot \exp_r(k^{(L-1)} g_L) \tag{7.29}$$

Moreover let

$$\hat{g}^{(L)} := \sum_{i=1}^{L} g_i \, h^{(i+1)}, \quad \text{with} \quad h^{(i)} := k_i^{-1} \cdot \dots \cdot k_L^{-1} \quad \text{for } i \leq L \text{ and} \quad h^{(L+1)} = 1, \tag{7.30}$$

then

$$\exp_r(\hat{g}^{(L)}) = \exp_r(g_1 \, h^{(2)}) \cdot \dots \cdot \exp_r(g_{L-1} \, h^{(L)}) \cdot \exp_r(g_L) \tag{7.31}$$

See [10] for the proof.

The exponential symmetry $S^+_{(L)}$

In this Section we identify the symmetry that will be used in the construction of the process ASIP(q,k). To have a symmetry that has quasi-product form over the sites we preliminary define more convenient generators of the $\mathscr{U}_q(\mathfrak{su}(1,1))$ quantum Lie algebra. Let

$$E := q^{K^0} K^+, \qquad F := K^- q^{-K^0} \qquad \text{and} \qquad K := q^{2K^0} \tag{7.32}$$

From the commutation relations (7.1) we deduce that (E, F, K) verify the relations

$$KE = q^2 EK \qquad \text{and} \qquad KF = q^{-2} FK \qquad [E,F] = -\frac{K - K^{-1}}{q - q^{-1}} \ . \tag{7.33}$$

Moreover, from Theorem 7.1, the following co-products

$$\Delta(E_1) := \Delta(q^{K_1^0}) \cdot \Delta(K_1^+) = E_1 \otimes \mathbf{1} + K_1 \otimes E_2 \tag{7.34}$$

$$\Delta(F_1) := \Delta(K_1^-) \cdot \Delta(q^{-K_1^0}) = F_1 \otimes K_2^{-1} + \mathbf{1} \otimes F_2 \tag{7.35}$$

are still symmetries of $H_{(2)}$. In general we can extend (7.34) and (7.35) to L sites, then we have that

$$\begin{aligned} E^{(L)} &:= \Delta^{(L-1)}(E_1) \\ &= \Delta^{(L-1)}(q^{K_1^0}) \cdot \Delta^{(L-1)}(K_1^+) \\ &= q^{K_1^0} K_1^+ + q^{2K_1^0 + K_2^0} K_2^+ + \dots + q^{2\sum_{i=1}^{L-1} K_i^0 + K_L^0} K_L^+ \\ &= E_1 + K_1 E_2 + K_1 K_2 E_3 + \dots + K_1 \cdot \dots \cdot K_{L-1} E_L \end{aligned} \tag{7.36}$$

$$
\begin{aligned}
F^{(L)} &:= \Delta^{(L-1)}(F_1) \\
&= \Delta^{(L-1)}(K_1^-)\cdot\Delta^{(L-1)}(q^{-K_1^0}) \\
&= K_1^- q^{-K_1^0-2\sum_{i=2}^L K_i^0} + \cdots + K_{L-1}^- q^{-K_{L-1}^0-2K_L^0} + K_L^- q^{-K_L^0} \\
&= F_1\cdot K_2^{-1}\cdot\ldots\cdot K_L^{-1} + \cdots + F_{L-1}\cdot K_L^{-1} + F_L
\end{aligned} \tag{7.37}
$$

are symmetries of H. If we consider now the symmetry obtained by q-exponentiating $E^{(L)}$ then this operator will pseudo-factorize by Proposition 7.2.

Lemma 7.1. *The operator*

$$
S_{(L)}^+ := \exp_{q^2}(E^{(L)}) \tag{7.38}
$$

is a symmetry of $H_{(L)}$. Its matrix elements are given by

$$
\langle \eta_1,...,\eta_L|S_{(L)}^+|\xi_1,...,\xi_L\rangle = \prod_{i=1}^L \sqrt{\binom{\eta_i}{\xi_i}_q \binom{\eta_i+2k-1}{\xi_i+2k-1}_q}\cdot \mathbf{1}_{\eta_i\geq\xi_i}\, q^{(\eta_i-\xi_i)\left[1+k+\xi_i+2\sum_{m=1}^{i-1}(\xi_m+k)\right]} \tag{7.39}
$$

Proof. From (7.33) we know that the operators E_i, K_i, copies of the operators defined in (7.32), verify the conditions (7.27) with $r=q^2$. As a consequence, from (7.36), (7.38) and Proposition 7.2, we have

$$
\begin{aligned}
S_{(L)}^+ &= \exp_{q^2}(E^{(L)}) \\
&= \exp_{q^2}(E_1)\cdot\exp_{q^2}(K_1E_2)\cdots\exp_{q^2}(K_1\cdots K_{L-1}E_L) \\
&= \exp_{q^2}\left(q^{K_1^0}K_1^+\right)\cdot\exp_{q^2}\left(q^{2K_1^0}q^{K_2^0}K_2^+\right)\cdots\exp_{q^2}\left(q^{2\sum_{i=1}^{L-1}K_i^0+K_L^0}K_L^+\right) \\
&= S_1^+S_2^+\cdots S_L^+
\end{aligned} \tag{7.40}
$$

where $S_i^+ := \exp_{q^2}\left(q^{2\sum_{m=1}^{i-1}K_m^0+K_i^0}K_i^+\right)$ has been defined. Using (7.26), we find

$$
S_i^+|\xi_1,\ldots,\xi_L\rangle = \sum_{\ell_i\geq 0}\frac{1}{[\ell_i]_q!}\left(q^{2\sum_{m=1}^{i-1}K_m^0+K_i^0}K_i^+\right)^{\ell_i} q^{-\frac{1}{2}\ell_i(\ell_i-1)}|\xi_1,\ldots,\xi_L\rangle \tag{7.41}
$$

$$
=\sum_{\ell_i\geq 0}\sqrt{\binom{\xi_i+\ell_i}{\ell_i}_q\cdot\binom{\xi_i+2k+\ell_i-1}{\ell_i}_q}\cdot q^{\ell_i(\xi_i+k+1)+2\ell_i\sum_{m=1}^{i-1}(\xi_m+k)}|\xi_1,\ldots,\xi_i+\ell_i,\ldots,\xi_L\rangle
$$

where in the last equality we used (7.4). Thus we find

$$
\begin{aligned}
S_{(L)}^+|\xi_1,\ldots,\xi_L\rangle &= S_1^+S_2^+\ldots S_L^+|\xi_1,\ldots,\xi_L\rangle \\
&= \sum_{\ell_1,\ell_2,\ldots,\ell_L\geq 0}\prod_{i=1}^L\left(\sqrt{\binom{\xi_i+\ell_i}{\ell_i}_q\cdot\binom{\xi_i+2k+\ell_i-1}{\ell_i}_q}\right. \\
&\qquad\left.\cdot\, q^{\ell_i(\xi_i+k+1)+2\ell_i\sum_{m=1}^{i-1}(\xi_m+k)}\right)|\xi_1+\ell_1,\ldots,\xi_L+\ell_L\rangle
\end{aligned} \tag{7.42}
$$

form which the matrix elements in (7.39) are immediately found. □

Construction of a positive ground state and the associated Markov process ASEP(q, j)

By applying Theorem 7.2 we are now ready to identify the stochastic process related to the Hamiltonian $H_{(L)}$ in (7.12).
We start from the state $|\mathbf{0}\rangle = |0, \ldots, 0\rangle$ which is obviously a trivial ground state of $H_{(L)}$. We then produce a positive ground state by acting with the symmetry $S^+_{(L)}$ in (7.38). Using (7.42) we obtain

$$|g\rangle \;=\; S^+_{(L)}|0, \ldots, 0\rangle = \sum_{\ell_1,\ell_2,\ldots,\ell_L \geq 0} \prod_{i=1}^{L} \sqrt{\binom{2k+\ell_i-1}{\ell_i}_q} \cdot \, q^{\ell_i(1-k+2ki)} \, |\ell_1, ..., \ell_L\rangle$$

Following the scheme in Theorem 7.2 we construct the operator $G_{(L)}$ defined by

$$G_{(L)}|\eta_1, \ldots, \eta_L\rangle = |\eta_1, \ldots, \eta_L\rangle\langle\eta_1, \ldots, \eta_L|S^+|0, \ldots, 0\rangle \tag{7.43}$$

In other words $G_{(L)}$ is represented by a diagonal matrix whose coefficients in the standard basis read

$$\langle\eta_1, \ldots, \eta_L|G_{(L)}|\xi_1, \ldots, \xi_L\rangle = \prod_{i=1}^{L} \sqrt{\binom{\eta_i+2k-1}{\eta_i}_q} \cdot q^{\eta_i(1-k+2ki)} \cdot \delta_{\eta_i=\xi_i} \tag{7.44}$$

Note that $G_{(L)}$ is factorized over the sites, i.e.

$$\langle\eta_1, \ldots, \eta_L|G_{(L)}|\xi_1, \ldots, \xi_L\rangle = \otimes_{i=1}^{L}\langle\eta_i|G_i|\xi_i\rangle \tag{7.45}$$

As a consequence of item a) of Theorem 7.2, the operator $\mathscr{L}^{(L)}$ conjugated to $H_{(L)}$ via $G^{-1}_{(L)}$, i.e.

$$\mathscr{L}^{(L)} = G^{-1}_{(L)} H_{(L)} G_{(L)} \tag{7.46}$$

is the generator of a Markov jump process $\eta(t) = (\eta_1(t), \ldots, \eta_L(t))$ describing particles jumping on the chain Λ_L. The state space of such a process is given by Ω_L and its elements are denoted by $\eta = (\eta_1, \ldots, \eta_L)$, where η_i is interpreted as the number of particles at site i. The asymmetry is controlled by the parameter $0 < q \leq 1$.

Proposition 7.3. *The action of the Markov generator $\mathscr{L}^{(L)} := G^{-1}_{(L)} H_{(L)} G_{(L)}$ is given by*

$$(\mathscr{L}^{(L)}f)(\eta) = \sum_{i=1}^{L-1} (\mathscr{L}_{i,i+1}f)(\eta) \qquad \textit{with}$$

$$\begin{aligned}(\mathscr{L}_{i,i+1}f)(\eta) \;&=\; q^{\eta_i-\eta_{i+1}+(2k-1)}[\eta_i]_q[2k+\eta_{i+1}]_q(f(\eta^{i,i+1}) - f(\eta)) \\ &+\; q^{\eta_i-\eta_{i+1}-(2k-1)}[2k+\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i}) - f(\eta))\end{aligned} \tag{7.47}$$

Proof. From Proposition 7.1 we know that $H^*_{(L)} = H_{(L)}$, hence we have that the operator $\tilde{H}_{(L)} := G_{(L)} H_{(L)} G^{-1}_{(L)}$ is the transposed of the generator $\mathscr{L}^{(L)}$ defined by (7.46). Then we have to verify that the transition rates to move from η to ξ for the Markov process generated by (7.47) are equal to the elements $\langle\xi|\tilde{H}_{(L)}|\eta\rangle$.

Since we already know that $\mathscr{L}^{(L)}$ is a Markov generator, in order to prove the result it is sufficient to apply the similarity transformation given by the matrix $G_{(L)}$ defined in (7.44) to the non-diagonal terms of (7.16), i.e. $q^{K_i^0} K_i^{\pm} K_{i+1}^{\mp} q^{-K_{i+1}^0}$. We show here the computation only for the first term, being the computation for the other term similar.
We have

$$\begin{aligned} &\langle \xi_i, \xi_{i+1} | G_i G_{i+1} \cdot q^{K_i^0} K_i^+ K_{i+1}^- q^{-K_{i+1}^0} \cdot G_i^{-1} G_{i+1}^{-1} | \eta_i, \eta_{i+1} \rangle \\ &= \langle \xi_i | G_i q^{K_i^0} K_i^+ G_i^{-1} | \eta_i \rangle \otimes \langle \xi_{i+1} | G_{i+1} K_{i+1}^- q^{-K_{i+1}^0} G_{i+1}^{-1} | \eta_{i+1} \rangle \end{aligned} \tag{7.48}$$

Using (7.44) and (7.4) one has

$$\langle \xi_i | G_i q^{K_i^0} K_i^+ G_i^{-1} | \eta_i \rangle = q^{\eta_i + 2 + 2ki} \, [2k + \eta_i]_q \langle \xi_i | \eta_i + 1 \rangle \tag{7.49}$$

and

$$\langle \xi_{i+1} | G_{i+1} K_{i+1}^- q^{-K_{i+1}^0} G_{i+1}^{-1} | \eta_{i+1} \rangle = q^{-\eta_{i+1} - 2k - 1 - 2ki} \, [\eta_{i+1}]_q \langle \xi_{i+1} | \eta_{i+1} - 1 \rangle \tag{7.50}$$

Multiplying the last two expressions one has

$$\langle \eta^{i+1,i} | \tilde{H}_{(L)} | \eta \rangle = q^{\eta_i - \eta_{i+1} - 2k + 1} [2k + \eta_i]_q [\eta_{i+1}]_q \tag{7.51}$$

that corresponds indeed to the rate to move from η to $\eta^{i+1,i}$ in (7.47). This concludes the proof. □

REMARK 7.5. *From item c) of Theorem 7.2, we have that the product measure $\mu_{(L)}$ defined by*

$$\mu_{(L)}(\eta) = \langle \eta | G_{(L)}^2 | \eta \rangle \tag{7.52}$$

is a reversible measure of $\mathscr{L}^{(L)}$. Notice that it corresponds to the reversible measure $\mathbb{P}^{(\alpha)}$ defined in (2.7) with the choice $\alpha = 1$.

7.3 Self-Duality of ASIP(q, k)

The following Proposition has been proven in [10] and it will be key to the proof of ASIP$(q, k$ self-duality.

PROPOSITION 7.4. *Let $A = A^*$ be a matrix with non-negative off-diagonal elements, and g an eigenvector of A with eigenvalue zero, with strictly positive entries. Let $\mathscr{L} = G^{-1} A G$ be the corresponding Markov generator. Let S be a symmetry of A, then $G^{-1} S G^{-1}$ is a self-duality function for the process with generator $\mathscr{L}$.*

We now use Proposition 7.4 and the exponential simmetry obtained in Section 7.2 to deduce a non-trivial duality function for the ASIP(q, k) process.

PROOF OF (5.2) **IN THEOREM 5.1**. From Proposition 7.1 we know that $H_{(L)}$ is self-adjoint, then, using Proposition 7.4 with $A = H_{(L)}$, $G = G_{(L)}$ given by (7.44) and $S = S_{(L)}^+$ given by (7.39) it follows that

$$G_{(L)}^{-1} S_{(L)}^+ G_{(L)}^{-1} \tag{7.53}$$

is a self-duality function for the process generated by $\mathscr{L}^{(L)}$. Its elements are computed as follows:

$$\begin{aligned}
&\langle\eta|G_{(L)}^{-1}S_{(L)}^{+}G_{(L)}^{-1}|\xi\rangle = \qquad (7.54)\\
&=\prod_{i=1}^{L}\left(\sqrt{\binom{2k+\eta_i-1}{\eta_i}_q}\cdot q^{\eta_i(1-k+2ki)}\right)^{-1}\langle\eta|S_i^{+}|\xi\rangle\left(\sqrt{\binom{2k+\xi_i-1}{\xi_i}_q}\cdot q^{\xi_i(1-k+2ki)}\right)^{-1}=\\
&=\prod_{i=1}^{L}\sqrt{\binom{\eta_i}{\xi_i}_q\binom{\eta_i+2k-1}{\xi_i+2k-1}_q\Big/\binom{2k+\eta_i-1}{\eta_i}_q\binom{2k+\xi_i-1}{\xi_i}_q}\,\cdot\\
&\qquad\cdot q^{(\eta_i-\xi_i)\left[2\sum_{m=1}^{i-1}(\xi_m+k)+\xi_i+k+1\right]-(2ki-k+1)(\eta_i+\xi_i)}\cdot\mathbf{1}_{\xi_i\le\eta_i}=\\
&=q^{2\sum_{i=1}^{L}(k\xi_i-\eta_i)}\prod_{i=1}^{L}\frac{[2k-1]_q![\eta_i]_q!}{[\xi_i+2k-1]_q![\eta_i-\xi_i]_q!}\cdot q^{(\eta_i-\xi_i)\left[2\sum_{m=1}^{i-1}\xi_m+\xi_i\right]-4ki\xi_i}\cdot\mathbf{1}_{\xi_i\le\eta_i}
\end{aligned}$$

Since both the original process and the dual process conserve the total number of particles it follows that $D_{(L)}$ in (5.2) is also a duality function. □

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