The spatial fluctuation theorem

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For systems of interacting particles and for interacting diffusions in d dimensions, driven out-of-equilibrium by an external field, a fluctuation relation for the generating function of the current is derived as a consequence of *spatial symmetries.* Those symmetries are in turn associated to transformations on the physical space that leave invariant the path space measure of the system without driving. This shows that in dimension $d \geq 2$ new fluctuation relations arise beyond the Gallavotti-Cohen fluctuation theorem related to the time-reversal symmetry.

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Background. The understanding of non-equilibrium systems has been a main goal of statistical physics over the last decades, where a remarkable progress has been reached. Key to this advance has been the study of the statistics of several observables ranging from the entropy production or the work, to the density or the current (of particle, charge, energy, momentum...). As the probability distribution of these observables follow a large deviation principle, the study of their corresponding large deviation functions (LDFs) has become a subject of primary interest, because LDFs can be considered as the *free energy* analog in non-equilibrium systems [1, 2]. For instance, the Macroscopic Fluctuation Theory [3], provides a variational formula from which one can compute the LDF of joint space-time densities and currents for driven diffusive systems, just by knowing two transport coefficients.

Particularly important in the understanding of non-equilibrium large deviations are the so called *fluctuations relations* [4–15], valid arbitrarily far from equilibrium. Those relations basically exploit the *time reversal symmetry* of the microscopic dynamics, that gives rise to a relation between a positive fluctuation of certain observable and the corresponding negative fluctuation. However, for systems endowed with a spatial structure one may wonder if *spatial symmetry* yields relations between the probabilities of vectorial observables in different directions. This was pointed out in [16], where an Isometric Fluctuation Relation (IFR) was obtained within the Macroscopic Fluctuation Theory framework under some assumptions and verified using numerical simulations. The IFR relation allows us to relate in a very simple manner any pair of isometric current fluctuations and it has been recently extended to the context of anisotropic systems [17].

Fluctuations relations are very relevant because from them one can derive in the linear response regime the Onsager reciprocity relations and the Green-Kubo formulas and, even more important, other reciprocity relations beyond these two can be obtained by considering higher order response terms [18]. In a similar way the IFR relation implies a set of new hierarchies between current cumulants and non-linear response coefficients [16].

The focus of this paper is on deriving *spatial fluctuation theorems* starting from the underlying stochastic microscopic dynamics. Whereas a clear understanding has been reached for the microscopic origin of the standard fluctuation theorem [5–8], it is an open problem to relate spatial fluctuation theorems to invariance properties of the microscopic dynamics. To achieve the result we will employ the Gibbs-formalism in space-time as introduced in [8] (see also [19] for the analysis *at equilibrium*).

Discrete systems. We first consider particle systems embedded in $\mathbb{Z}^d$ following a Markov dynamics. For the sake of simplicity we restrict to discrete time, however similar results hold true for continuos time dynamics. The Markov process is denoted by $\{X(n) = (X_k(n))_{k\in\mathbb{N}} : n \in \mathbb{N}\}$, where $X_k(n)$ denotes the random position of k^{th} particle at time n. System configurations are denoted by the vector $x = (x_1, x_2, \dots)$ with the first particle located at site $x_1 \in \Lambda_L = \{1, \dots, L\}^d \subseteq \mathbb{Z}^d$ and similarly for all the other particles. During the evolution particles jump to their nearest neighbor sites with anisotropic probabilities. The anisotropy is tuned by the d-dimensional vector $a = (a_1, \dots, a_d)$ and in each direction we assume a weak asymmetry which is produced by a constant external field $E = (E_1, \dots, E_d)$. The dynamics is defined by the probability transition matrix with elements

$$p(x, y) = \begin{cases} b(x, y) a_s e^{\pm \frac{E_s}{L}} & \text{if } y = x^{k,\pm e_s} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for $k \in \mathbb{N}$ and $s = 1, \dots, d$. Here e_s is the unit vector in the s-direction and $x^{k,\pm e_s}$ is the configuration that is obtained from the configuration x by moving the k^{th} particle to $x_k \pm e_s$. We define $p_0(x, y) := b(x, y)a_s$ as the jump probabilities of the symmetric system (i.e. $E = 0$).

We consider an initial particle number (a conserved quantity of the dynamics) proportional to the volume, i.e. $N = \rho L^d$, corresponding to a constant density $0 < \rho < \infty$. For $s = 1, \dots, d$, we denote by $Q^{i,s}_{M,L}$ the *microscopic currents* (net number of particles) across the bond $(i, i+$

e_s) in the volume L^d up to time M. Namely,

$$Q^{i,s}_{M,L} = \sum_{n=0}^{M-1} \sum_{k\geq 1} (\mathbf{1}_{\{X(n+1)=X(n)^{k,e_s}\}} - \mathbf{1}_{\{X(n+1)=X(n)^{k,-e_s}\}})$$

where $\mathbf{1}_{\{A\}}$ is the indicator function of the set A. We collect them into the vector $Q^i_{M,L} = (Q^{i,1}_{M,L}, \dots, Q^{i,d}_{M,L})$ and we will be interested in the large deviations of the current vector $Q_{M,L} = \frac{1}{L}\sum_{i\in\Lambda_L} Q^i_{M,L}$. This current is an additive functional of the Markov process $(X(n))_{n\in\mathbb{N}}$ and it satisfies a large deviation principle that can be informally stated as $\mathbb{P}(Q_{M,L} \sim Mq) \approx e^{-MI_L(q)}$ as $M \to \infty$. By the Gartner-Ellis theorem [20] the large deviation function I_L can be obtained as $I_L(q) = \sup_\lambda(\lambda\cdot q - \mu_L(\lambda))$ where we define the finite-volume scaled cumulant generating function of the current as $\mu_L(\lambda) := \lim_{M\to\infty} \frac{1}{M} \ln Z_{M,L}(\lambda)$, with $Z_{M,L}(\lambda) = \mathbb{E}(e^{\lambda\cdot Q_{M,L}})$, where $\mathbb{E}(\cdot)$ denotes the average in path space. The application of Perron-Frobenius theorem allow us to express $\mu_L(\lambda)$ as the logarithm of the largest eigenvalue of the tilted matrix with elements $p_\lambda(x, y) = b(x, y)a_s e^{\pm\left(\frac{E_s}{L}+\frac{\lambda_s}{L}\right)}$ if $y = x^{k,\pm e_s}$ and $p_\lambda(x, y) = 0$ otherwise. To study the system in the thermodynamic limit one needs to rescale by defining

$$\mu(\lambda) := \lim_{L\to\infty} \frac{\mu_L(\lambda)}{L^{d-2}} = \lim_{L\to\infty}\lim_{M\to\infty} \frac{\ln Z_{M,L}(\lambda)}{L^{d-2}M}. \qquad (2)$$

Our first result provides the derivation of an *anisotropic fluctuation relation* in d dimensions from the assumption of a system invariance property.

Theorem (Spatial fluctuation theorem). *Consider N particles whose dynamics is defined by the transition matrix* (1). *Let P denote the probability measure on the path space Ω in the presence of an external field $E = (E_1, \dots, E_d)$. Consider a transformation on the physical space $U : \mathbb{Z}^d \to \mathbb{Z}^d$ and, for a trajectory in path space $\vec{x} = (x(0), x(1), \dots x(M))$, consider the bijective mapping $\mathscr{R} : \Omega \to \Omega$ induced by U*

$$(\mathscr{R}x(n))_k = Ux_k(n) \qquad (3)$$

such that

$$Q_{M,L}(\mathscr{R}\vec{x}) = U\, Q_{M,L}(\vec{x}). \qquad (4)$$

Assume that P_0, the measure of the symmetric system with $E = 0$, has the invariance property

$$P_0(\vec{x}) = P_0(\mathscr{R}\vec{x}) \qquad \forall\, \vec{x} \in \Omega\,. \qquad (5)$$

Then the following fluctuation relations hold: $\forall\lambda$

$$Z_{M,L}(\lambda) = Z_{M,L}\Big((U^{-1})^t(\lambda + E) - E\Big), \qquad (6)$$

$$I_L(q) - I_L(Uq) = (U^tE - E)\cdot q \qquad (7)$$

where $(U^{-1})^t$ denotes the transposed of the inverse.

Notice that statement (6) is obtained at finite time and finite volume, whereas eq. (7) holds for any finite volume in the limit of large times. Furthermore, when the limit in (2) is finite, one has $\mathbb{P}(Q_{M,L} \sim ML^{d-2}q) \approx e^{-ML^{d-2}I(q)}$ with $I(q) = \lim_{L\to\infty} I_L(q)/L^{d-2}$ satisfying the analogous of eq. (7). We remark as well that to satisfy relation (5) the map U will also depend on the volume L and the anisotropy a. However, to alleviate notation we do not write this dependence explicitly. In the proof we also shorthand $Z_{M,L}(\lambda)$ as $Z(\lambda)$ and $Q_{M,L}(\vec{x})$ as $Q(\vec{x})$.

Proof. We start by observing that, from the previous definition of the current the following relation holds

$$P(\vec{x}) = \exp\left[E\cdot Q(\vec{x})\right]P_0(\vec{x}). \qquad (8)$$

Using the invariance property (5) in the hypothesis of the theorem and applying the relation (8) we then have

$$Z(\lambda) = \sum_{\vec{x}\in\Omega} P(\vec{x})e^{\lambda\cdot Q(\vec{x})} = \sum_{\vec{x}\in\Omega} P(\mathscr{R}\vec{x})e^{-E\cdot Q(\mathscr{R}\vec{x})+(E+\lambda)\cdot Q(\vec{x})}$$

Applying the change of variables $\vec{y} = \mathscr{R}\vec{x}$ and since $\mathscr{R}$ is a bijective map we find

$$Z(\lambda) = \sum_{y\vec{\in}\Omega} P(\vec{y})e^{-E\cdot Q(\vec{y})+(E+\lambda)\cdot Q(\mathscr{R}^{-1}\vec{y})}.$$

Hence, using the assumption (4), it is easy to check that (6) follows. By taking the limit $M \to \infty$ the same relation holds for μ_L and therefore (7) follows by Legendre transform. ☐

Comments and examples. If the transformation U is chosen as spatial inversion, i.e. $Ui = -i$ for $i \in \mathbb{Z}^d$, then one recovers the standard Gallavotti-Cohen fluctuation relation, i.e. $\mu_L(\lambda) = \mu_L(-\lambda - 2E)$. Notice that usually this relation is associated to time reversal invariance of the measure of the symmetric system, namely $P_0(\vec{x}) = P_0(\mathscr{T}\vec{x})$ where $\mathscr{T}$ is the transformation that produces time reversal, i.e. for a given path $\vec{x} = (x(0), x(1), \dots, x(M))$ one defines $\mathscr{T}\vec{x} = (x(M), x(M-1), \dots, x(0))$. It was remarked in [7] that any transformation on path space such that $\mathscr{R}\circ\mathscr{R} = 1$ would lead to the Gallavotti-Cohen fluctuation relation. The transformation on path space induced by spatial inversion in physical space has indeed such property.

More generally, the theorem above allows us to deduce generalised fluctuation relations as a consequence of *spatial symmetries*, i.e. whenever a transformation U on the physical space satisfies (4) and (5) then (6) and (7) follow. To further illustrate this point we shall discuss examples of systems of non-interacting particles. In this case the dynamics can be studied in terms of a single particle and the scaled cumulant generating function μ_L can be explicitly solved, so that one can check by inspection which spatial symmetries hold. For instance, for a system of independent random walkers (RW) where each particle at site i jumps to site $i \pm e_s$ with probability $a_s e^{\pm\frac{E_s}{L}}$ with periodic boundary conditions, an elementary application of Perron-Frobenius theorem gives

$$\mu_L^{\mathrm{RW}}(\lambda) = \rho L^d \ln\left[\sum_{s=1}^{d}\left(a_s e^{(E_s+\lambda_s)/L} + a_s e^{-(E_s+\lambda_s)/L}\right)\right].$$

For $d = 2$ and by doing a change of variables to polar coordinates $(z,\,\theta)$ such that $\lambda_1 = z\cos\theta - E_1$ and $\lambda_2 = z\sin\theta\sqrt{\alpha} - E_2$, with $\alpha = a_1/a_2$ being the anisotropy ratio, the above expression reads

$$\mu_L^{\mathrm{RW}}(z,\theta) = \rho L^2 \ln\left[a_1\left(e^{\frac{z\cos\theta}{L}} + e^{\frac{-z\cos\theta}{L}}\right)\right. \\ \left. + a_2\left(e^{\frac{z\sqrt{\alpha}\sin\theta}{L}} + e^{\frac{-z\sqrt{\alpha}\sin\theta}{L}}\right)\right]. \qquad (9)$$

In the isotropic case ($\alpha = 1$) one recognizes by inspection the *discrete symmetries* of the system leading to a fluctuation theorem. Namely, $\mu_L^{\mathrm{RW}}(z,\theta) = \mu_L^{\mathrm{RW}}(z,\theta')$, for $\theta' = m\pi/2 \pm \theta$, $\forall m \in \mathbb{Z}$ (see Fig. 1). It is then natural to expect that for diffusions one is lead to consider *continuous spatial symmetries*. This can be seen by considering the diffusive scaling limit of the previous example, leading to a system of independent Brownian motions with drift. If we define $\{X^{(L)}(n) = (X_k^{(L)}(n))_{k\in\mathbb{N}} : n \in \mathbb{N}\}$ the positions at time n of the independent random walkers (labeled by k) in a volume of linear size L, then the process $\{R_t = (R_{k,t})_{k\in\mathbb{N}} : t \in \mathbb{R}^+\}$ defined by $R_t := \lim_{L\to\infty} \frac{X^{(L)}(\lfloor L^2 t\rfloor)}{L}$ will be a family of anisotropic independent Brownian motions with drift, satisfying the stochastic differential equation $dR_{k,t} = 2AEdt + \sqrt{A}dB_{k,t}$ with A the $d\times d$ diagonal matrix with elements $A_{s,s} = 2a_s$ and $B_{k,t} \in \mathbb{R}^d$ denoting the standard Brownian motion. An immediate computation gives

$$\lim_{L\to\infty} \frac{\mu_L^{RW}(\lambda)}{L^{d-2}} = \mu^{BM}(\lambda) = \rho\sum_{s=1}^{d} a_s\lambda_s(\lambda_s + 2E_s) \qquad (10)$$

where $\mu^{\mathrm{BM}}(\lambda) := \lim_{t\to\infty}\frac{1}{t}\ln\left[\mathbb{E}(e^{\lambda\cdot R_t})\right]$. From the explicit expression (10) we see that a spatial fluctuation relation holds, i.e.

$$\mu^{BM}(\lambda) = \mu^{BM}\left((U^{-1})^t(\lambda + E) - E\right) \qquad (11)$$

with U such that $UAU^t = A$. For $d = 2$, U takes the form given at the end of the caption of Fig. 1. As we shall see below, such a relation can be traced back to the invariance of the path measure of the anisotropic Brownian motion under a spatial transformation U such that $UAU^t = A$ (in agreement with the findings of [16, 17] in the context of the Macroscopic Fluctuation Theory [3]).

Diffusions. To state a (spatial) fluctuation relation for systems following a generic diffusion process we consider an abstract path space Ω, and a bijective measurable transformation $\mathscr{R} : \Omega \to \Omega$ with inverse $\mathscr{R}^{-1}$. Elements in the path space are denoted by ω. For diffusions we will

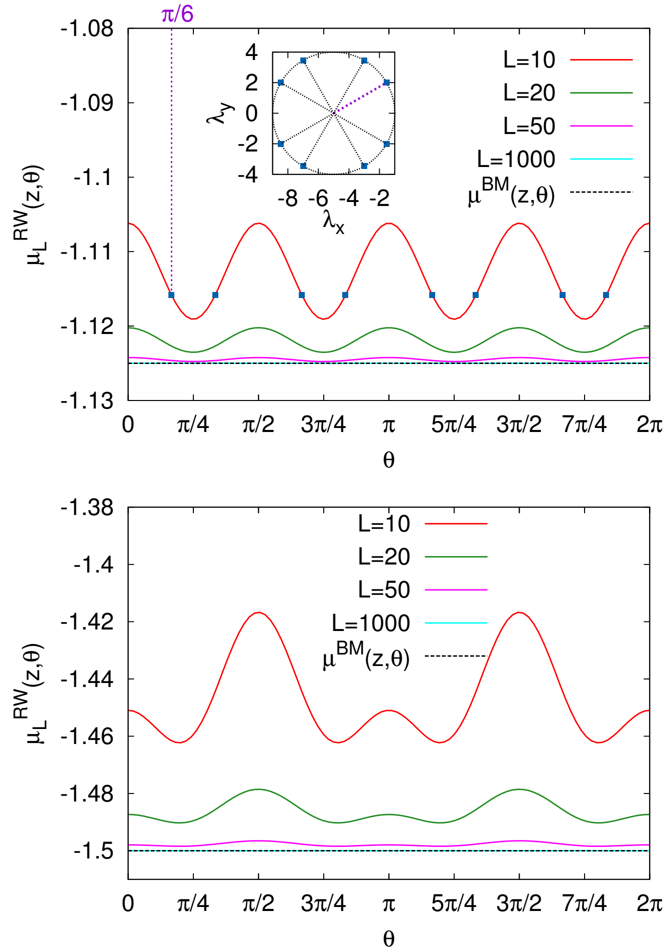

FIG. 1. Plot of $\frac{\mu_L^{RW}(z,\theta)}{L^{d-2}}$ in $d = 2$ as a function of $\theta = \arctan(\frac{1}{\sqrt{\alpha}}\frac{\lambda_2+E_2}{\lambda_1+E_1})$. In both figures $\rho = 0.5$, $E = (5,0)$, $z = 4$ and the anisotropy ratio is $\alpha = 1$ in the top and $\alpha = 2$ in the bottom. The inset of the top panel shows the 8 discrete symmetries of the system in λ-space for $L = 10$, starting from an initial angle e.g. $\theta = \pi/6$, corresponding to the transformations U given by the diagonal and the anti diagonal matrices with elements ± 1. As L increases the discrete symmetry scales to a continuous symmetry associated to the transformation $U = \begin{pmatrix} \cos\theta & -\sqrt{\alpha}\sin\theta \\ \frac{1}{\sqrt{\alpha}}\sin\theta & \cos\theta \end{pmatrix}$ and (11) is then satisfied.

consider $\Omega = \mathscr{C}([0,T],\mathbb{R}^d)$, i.e. the set of all continuos paths up to time T taking value in $\mathbb{R}^d$, and

$$(\mathscr{R}(\omega))_t = U\omega_t \qquad (12)$$

with $U : \mathbb{R}^d \to \mathbb{R}^d$ an invertible spatial transformation.

Proposition. *Consider a probability measure on path space Ω of the form*

$$P(d\omega) = e^{H(\omega)}P_0(d\omega) \qquad (13)$$

where P_0 is $\mathscr{R}$-invariant. For all $\varphi : \Omega \to \mathbb{R}$ we have the identity

$$\mathbb{E}(e^{\varphi\circ\mathscr{R}}) = \mathbb{E}(e^{\varphi}e^{H\circ\mathscr{R}^{-1} - H}) \qquad (14)$$

Proof.

$$\int e^{\varphi\circ\mathscr{R}}dP = \int e^{\varphi\circ\mathscr{R}}e^{H}dP_0 = \int e^{\varphi}e^{H\circ\mathscr{R}^{-1}}dP_0$$
$$= \int e^{\varphi}e^{H\circ\mathscr{R}^{-1}-H}dP$$

□

In the particular case that $\varphi = -\gamma(H\circ\mathscr{R}^{-1} - H)$ with $\gamma\in\mathbb{R}$, the relation (14) gives $\mathbb{E}(e^{\gamma(H\circ\mathscr{R}-H)}) = \mathbb{E}(e^{(1-\gamma)(H\circ\mathscr{R}^{-1}-H)})$. Even more, specifying to a transformation such that $\mathscr{R} = \mathscr{R}^{-1}$ this identity is exactly a symmetry of the form of the standard fluctuation theorem for the quantity $H\circ\mathscr{R} - H$.
Denoting by $P\circ\mathscr{R}$ the image measure of P under $\mathscr{R}$, i.e., $\int f d(P\circ\mathscr{R}) = \int (f\circ\mathscr{R})dP$, it can be readily verified that $dP = e^{H}dP_0$ implies that $d(P\circ\mathscr{R}) = e^{H\circ\mathscr{R}^{-1}}dP_0$. Therefore, as we shall use below, we can write

$$e^{H\circ\mathscr{R}^{-1}-H} = \frac{d(P\circ\mathscr{R})}{dP} = \frac{d(P\circ\mathscr{R})}{dP_0}\Big/\frac{dP}{dP_0}. \qquad (15)$$

The abstract setting of the proposition above can be used to derive the spatial fluctuation theorem for finite time T in the context of interacting diffusions. We shall illustrate this by considering the overdamped Langevin dynamics $\{X_t = (X_{k,t})_{k\in\mathbb{N}} :\ t\in\mathbb{R}^+\}$ describing N particles (labeled by k) subject to a drift vector which can arise from an applied force to each particle F and/or a conservative potential V, with a positive definite constant diffusion matrix A. Then the stochastic differential equation for the k^{th} particle reads

$$dX_{k,t} = F(X_{k,t})dt + \nabla_k V(X_t)dt + \sqrt{A}dB_{k,t} \qquad (16)$$

where $B_{k,t}$ is again a standard Brownian motion. Notice that V can model a self-potential as well as an interaction potential. To obtain the new process applying a transformation $\mathscr{R}$ of the type (12), we remind that if X is multivariate normal distributed with mean zero and covariance matrix A, then for any $d\times d$ matrix U, UX is multivariate normal distributed with mean zero and covariance UAU^t. As a consequence if $Y_{k,t} = UX_{k,t}$ then

$$dY_{k,t} = UF(U^{-1}Y_{k,t})dt + U\nabla_k V(U^{-1}Y_t)dt + \sqrt{UAU^t}dB_{k,t}$$

where we denote $U^{-1}Y_t$ the collection $U^{-1}Y_{k,t}$ $\forall k$. For the process Y_t to be absolutely continuous w.r.t. the X_t process, we need that $UAU^t = A$. Moreover, assuming that the potential is invariant under the transformation U, i.e. $V(Ux) = V(x)$, the process $Y_{k,t}$ satisfies

$$dY_{k,t} = UF(U^{-1}Y_{k,t})dt + \nabla_k V(Y_t)dt + \sqrt{A}dB_{k,t}. \qquad (17)$$

Thus, the process whose paths' distribution is invariant under the transformation $\mathscr{R}$ is

$$dZ_{k,t} = \nabla_k V(Z_t)dt + \sqrt{A}dB_{k,t} \qquad (18)$$

By using the Girsanov formula [21] one can compute $dP_X = e^{H}dP_Z$, i.e. the relative density between the path space measure P_X of the process (16) and the path space measure measure P_Z of the process (18). Analogously the measure P_Y of the process (17) and the measure P_Z are related by $d(P_X\circ\mathscr{R}) = dP_Y = e^{H\circ\mathscr{R}^{-1}}dP_Z$. Hence, if we denote $\tilde{U}$ and $\tilde{A}$ as the $N\times(d\times d)$ block diagonal matrices with elements U and A respectively on the diagonal and $\tilde{F}$ as the $N\times d$ column vector consisting in copying N times the vector F, then by applying (15) we get

$$\begin{aligned}(H\circ\mathscr{R}^{-1} - H)(\omega) &= \int_0^T \tilde{A}^{-1}(\tilde{U}\tilde{F}(\tilde{U}^{-1}\omega_t) - \tilde{F}(\omega_t))d\omega_t \\ &- \int_0^T \tilde{A}^{-1}(\tilde{U}F(\tilde{U}^{-1}\omega_t) - \tilde{F}(\omega_t))\cdot\nabla V(\omega_t)dt \\ &- \frac{1}{2}\int_0^T (\tilde{U}\tilde{F}(\tilde{U}^{-1}\omega_t)\cdot\tilde{A}^{-1}(\tilde{U}\tilde{F}(\tilde{U}^{-1}\omega_t)dt \\ &+ \frac{1}{2}\int_0^T \tilde{F}(\omega_t)\cdot\tilde{A}^{-1}\tilde{F}(\omega_t)dt. \qquad (19)\end{aligned}$$

In addition, if the force F is constant we find (we put $\omega_0 = 0$)

$$(H\circ\mathscr{R}^{-1} - H)(\omega) = A^{-1}(UF - F)\cdot Q_T(\omega) \qquad (20)$$

with
$$Q_T(\omega) = \sum_{k=1}^{N}\Big(\omega_{k,T} - \int_0^T \nabla_k V(\omega_t)dt\Big). \qquad (21)$$

Furthermore, by choosing $\varphi(\omega) = \lambda\cdot Q_T(\omega)$, with $\lambda\in\mathbb{R}^d$, we have that $(\varphi\circ\mathscr{R})(\omega) = \lambda\cdot UQ_T(\omega) = U^t\lambda\cdot Q_T(\omega)$ and from (14) and (20) we get $\mathbb{E}(e^{U^t\lambda\cdot Q_T}) = \mathbb{E}(e^{(\lambda + A^{-1}(UF-F)\cdot Q_T})$. By defining $Z_T(\lambda) := \mathbb{E}(e^{\lambda\cdot Q_T})$ it readily follows that

$$Z_T(\lambda) = Z_T\Big((U^{-1})^t(\lambda + A^{-1}F) - A^{-1}F\Big) \qquad (22)$$

which is the analogous relation to the previously found for the discrete setting (6) . Notice that if A is the diagonal matrix with elements $A_{s,s} = 2a_s$, $F = AE$, $V = 0$ and $\mu^{BM}(\lambda) := \lim_{T\to\infty} T^{-1}\ln Z_T(\lambda)$ we recover from the above equation the previous result (11) for the anisotropic Brownian motion with drift.

Conclusions: In this letter we have derived a *spatial fluctuation theorem* (SFT) for interacting particle systems and interacting diffusions driven out of equilibrium by an external field. It has been proved in both cases that the SFT can be traced back to an invariance property of the microscopic path space measure under spatial transformations. Remarkably, this result holds for finite time in the case of interacting diffusions (22), with $UAU^t = A$ and $V(Ux) = V(x)$, as well as for finite volume for interacting particle systems (6). In the latter, the spatial transformations yielding the SFT correspond to discrete symmetries associated with the underlying lattice geometry of the system at hand. However, as the system linear size increases these transformations become continuous, as is depicted in Fig. 1 for the particular case of non-interacting random walkers.

The SFT gives a new perspective on how the microscopic symmetries of the system are reflected at the fluctuating level. Whereas the standard fluctuation theorem is based on time reversal-symmetry of the microscopic dynamics, the SFT proves that the spatial symmetries have also a word to say. It is worth emphasizing that from the SFT new hierarchies for the currents cumulants and for the non-linear response can be obtained [16, 17]. In addition, from an experimental point of view (see e.g. [22, 23]), checking the SFT and its consequences would also be very interesting and statistically feasible since one does not have necessarily to measure the unlikely inverse fluctuation of the vectorial observable at hand, but just a spatial transformation of it. The study of the full Langevin dynamics (including inertia) also remains an open problem.

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