The spatial fluctuation theorem

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For systems of interacting particles and for interacting diffusions in d dimensions, driven out-of-equilibrium by an external field, a fluctuation relation for the generating function of the current is derived as a consequence of spatial symmetries. Those symmetries are in turn associated to transformations on the physical space that leave invariant the path space measure of the system without driving. This shows that in dimension d ≥ 2 new fluctuation relations arise beyond the Gallavotti-Cohen fluctuation theorem related to the time-reversal symmetry.

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Background. The understanding of non-equilibrium systems has been a main goal of statistical physics over the last decades, where a remarkable progress has been reached. Key to this advance has been the study of the statistics of several observables ranging from the entropy production or the work, to the density or the current (of particle, charge, energy, momentum...). As the probability distribution of these observables follow a large deviation principle, the study of their corresponding large deviations (LDFs) has become a subject of primary interest, because LDFs can be considered as the deviation functions (LDFs) has become a subject of priorization principle, the study of their corresponding large fluctuations relations are very relevant because from the focusing theorem [5–8], it is an open problem to relate spatial fluctuation theorems to invariance properties of the microscopic dynamics. To achieve the result we will employ the Gibbs-formalism in space-time as introduced in [8] (see also [19] for the analysis at equilibrium).

Discrete systems. We first consider particle systems embedded in Zd following a Markov dynamics. For the sake of simplicity we restrict to discrete time, however similar results hold true for continuous time dynamics. The Markov process is denoted by \{X(n) = (X_k(n))_{k∈N} : n ∈ N\}, where X_k(n) denotes the random position of k-th particle at time n. System configurations are denoted by the vector x = (x_1, x_2, ... ) with the first particle located at site x_1 ∈ Λ_L = \{1, ..., L\}^d ⊆ Zd and similarly for all the other particles. During the evolution particles jump to their nearest neighbor sites with anisotropic probabilities. The anisotropy is tuned by the d-dimensional vector a = (a_1, ..., a_d) and in each direction we assume a weak asymmetry which is produced by a constant external field F = (E_1, ..., E_d). The dynamics is defined by the probability transition matrix with elements

\begin{equation}
    p(x, y) = \begin{cases} 
        b(x, y)a_s e^{\pm \frac{F s}{k_B T}} & \text{if } y = x^{k, \pm e_s} \\
        0 & \text{otherwise},
    \end{cases}
\end{equation}

for k ∈ N and s = 1, ..., d. Here e_s is the unit vector in the s-direction and x^{k, \pm e_s} is the configuration that is obtained from the configuration x by moving the k-th particle to x_k ± e_s. We define p_0(x, y) := b(x, y)a_s as the jump probabilities of the symmetric system (i.e. F = 0).

We consider an initial particle number (a conserved quantity of the dynamics) proportional to the volume, i.e. \( N = pL^d \), corresponding to a constant density 0 < p < ∞. For s = 1, ..., d, we denote by Q_{j,M,L} the microscopic currents (net number of particles) across the bond (i,j+...
where $1_{\{A\}}$ is the indicator function of the set $A$. We collect them into the vector $Q_{M,L}^{i,s} = (Q_{M,L}^{i,1}, \ldots , Q_{M,L}^{i,d})$ and we will be interested in the large deviations of the current vector $Q_{M,L} = \frac{1}{L} \sum_{i \in \lambda, L} Q_{M,L}^{i,s}$. This current is an additive functional of the Markov process $(X(n))_{n \in \mathbb{N}}$ and it satisfies a large deviation principle that can be informally stated as $\mathbb{P}(Q_{M,L} \sim ML^{d-2}q) \approx e^{-ML^{d-2}I(q)}$ as $M \to \infty$. By the Gartner-Ellis theorem [20] the large deviation function $I_L$ can be obtained as $I_L(q) = \sup_{\lambda}(\lambda \cdot q - \mu_L(\lambda))$ where we define the finite-volume scaled cumulant generating functional of the current as $\mu_L(\lambda) := \lim_{M \to \infty} \frac{1}{ML} \ln Z_{M,L}(\lambda)$, with $Z_{M,L}(\lambda) = \mathbb{E}(e^{\lambda Q_{M,L}})$, where $\mathbb{E}(\cdot)$ denotes the average in path space. The application of Perron-Frobenius theorem allow us to express $\mu_L(\lambda)$ as the logarithm of the largest eigenvalue of the tilted matrix with elements $p_{\lambda}(x,y) = b(x,y)a_0 e^{\pm \lambda \frac{y}{L} \frac{E}{2}}$ if $y = x \pm e_s$ and $p_{\lambda}(x,y) = 0$ otherwise. To study the system in the thermodynamic limit one needs to rescale by defining

$$\mu(\lambda) := \lim_{L \to \infty} \frac{\mu_L(\lambda)}{L^{d-2}} = \lim_{L \to \infty} \frac{\ln Z_{M,L}(\lambda)}{L^{d-2}M}. \tag{2}$$

Our first result provides the derivation of an anisotropic fluctuation relation in $d$ dimensions from the assumption of a system invariance property.

**Theorem (Spatial fluctuation theorem).** Consider $N$ particles whose dynamics is defined by the transition matrix $[1]$. Let $P$ denote the probability measure on the path space $\Omega$ in the presence of an external field $E = (E_1, \ldots , E_d)$. Consider a transformation on the physical space $U : \mathbb{Z}^d \to \mathbb{Z}^d$ and, for a trajectory in path space $\vec{x} = (x(0), x(1), \ldots , x(M))$, consider the bijective mapping $\mathcal{R} : \Omega \to \Omega$ induced by $U$

$$(\mathcal{R}x(n))_k = Ux_k(n) \tag{3}$$

such that

$$Q_{M,L}(\mathcal{R}\vec{x}) = U Q_{M,L}(\vec{x}). \tag{4}$$

Assume that $P_0$, the measure of the symmetric system with $E = 0$, has the invariance property

$$P_0(\vec{x}) = P_0(\mathcal{R}\vec{x}) \quad \forall \vec{x} \in \Omega. \tag{5}$$

Then the following fluctuation relations hold: $\forall \lambda$

$$Z_{M,L}(\lambda) = Z_{M,L}\left((U^{-1})^t(\lambda + E) - E\right), \tag{6}$$

$$I_L(q) - I_L(Uq) = (U^t E - E) \cdot q \tag{7}$$

where $(U^{-1})^t$ denotes the transposed of the inverse.

Notice that statement (6) is obtained at finite time and finite volume, whereas eq. (7) holds for any finite volume in the limit of large times. Furthermore, when the limit in (7) is finite, one has $\mathbb{P}(Q_{M,L} \sim ML^{d-2}q) \approx e^{-ML^{d-2}I(q)}$ with $I(q) = \lim_{L \to \infty} I_L(q)/L^{d-2}$ satisfying the analogous of eq. (7). We remark as well that to satisfy relation (5) the map $U$ will also depend on the volume $L$ and the anisotropy $a$. However, to alleviate notation we do not write this dependence explicitly. In the proof we also shorthand $Z_{M,L}(\lambda)$ as $Z(\lambda)$ and $Q_{M,L}(\vec{x})$ as $Q(\vec{x})$.

**Proof.** We start by observing that, from the previous definition of the current the following relation holds

$$P(\vec{x}) = \exp \{E \cdot Q(\vec{x})\} P_0(\vec{x}). \tag{8}$$

Using the invariance property (5) in the hypothesis of the theorem and applying the relation (7) we then have

$$Z(\lambda) = \sum_{\vec{x} \in \Omega} P(\vec{x}) e^{\lambda Q(\vec{x})} = \sum_{\vec{x} \in \Omega} P(\mathcal{R}\vec{x}) e^{-E \cdot Q(\mathcal{R}\vec{x})} + (E + \lambda) \cdot Q(\vec{x})$$

Applying the change of variables $\vec{y} = \mathcal{R}\vec{x}$ and since $\mathcal{R}$ is a bijective map we find

$$Z(\lambda) = \sum_{\vec{y} \in \Omega} P(\vec{y}) e^{-E \cdot Q(\vec{y})} + (E + \lambda) \cdot Q(\mathcal{R}^{-1}\vec{y}).$$

Hence, using the assumption (4), it is easy to check that (6) follows. By taking the limit $M \to \infty$ the same relation holds for $\mu_L$ and therefore (7) follows by Legendre transform.

**Comments and examples.** If the transformation $U$ is chosen as spatial inversion, i.e. $U_i = -i$ for $i \in \mathbb{Z}^d$, then one recovers the standard Gallavotti-Cohen fluctuation relation, i.e. $\mu_L(\lambda) = \mu_L(-\lambda - 2E)$. Notice that usually this relation is associated to time reversal invariance of the measure of the symmetric system, namely $P_0(\vec{x}) = P_0(\mathcal{S}\vec{x})$ where $\mathcal{S}$ is the transformation that produces time reversal, i.e. for a given path $\vec{x} = (x(0), x(1), \ldots , x(M))$ one defines $\mathcal{S}\vec{x} = (x(M), x(M-1), \ldots , x(0))$. It was remarked in [7] that any transformation on path space such that $\mathcal{R} \circ \mathcal{R} = 1$ would lead to the Gallavotti-Cohen fluctuation relation. The transformation on path space induced by spatial inversion in physical space has indeed such property.

More generally, the theorem above allows us to deduce generalised fluctuation relations as a consequence of spatial symmetries, i.e. whenever a transformation $U$ on the physical space satisfies (4) and (6) then (4) and (7) follow. To further illustrate this point we shall discuss examples of systems of non-interacting particles. In this case the dynamics can be studied in terms of a single particle and the scaled cumulant generating function $\mu_L$ can be explicitly solved, so that one can check by inspection which spatial symmetries hold. For instance, for a system of independent random walkers (RW) where each particle at site $i$ jumps to site $i \pm e_s$ with probability $a_s e^{\pm \frac{E_s}{L}}$ with
periodic boundary conditions, an elementary application of Perron-Frobenius theorem gives
\[
\mu_{RW}^L(\lambda) = \rho L^d \ln \left[ \sum_{s=1}^{d} \left( a_s e^{(E_s + \lambda_s)/L} + a_s e^{-(E_s + \lambda_s)/L} \right) \right].
\]

For \( d = 2 \) and by doing a change of variables to polar coordinates \((z, \theta)\) such that \( \lambda_1 = z \cos \theta - E_1 \) and \( \lambda_2 = z \sin \theta \sqrt{\alpha} - E_2 \), with \( \alpha = a_1/a_2 \) being the anisotropy ratio, the above expression reads
\[
\mu_{RW}^L(z, \theta) = \rho L^2 \ln \left[ \left( a_1 \left( e^{\pi \cos \theta/\ell} + e^{-\pi \cos \theta/\ell} \right) \right)
+ a_2 \left( e^{\pi \sin \theta/\ell} + e^{-\pi \sin \theta/\ell} \right) \right]. \tag{9}
\]

In the isotropic case \( (\alpha = 1) \) one recognizes by inspection the discrete symmetries of the system leading to a fluctuation theorem. Namely, \( \mu_{RW}^L(z, \theta) = \mu_{RW}^L(z, \theta') \), for \( \theta' = m \pi/2 \pm \theta \), \( \forall m \in \mathbb{Z} \) (see Fig. 1). It is then natural to expect that for diffusions one is lead to consider continuous spatial symmetries. This can be seen by considering the diffusive scaling limit of the previous example, leading to a system of independent Brownian motions with drift. If we define \( \{X^{(L)}(n) = (X^{(L)}_k(n))_{k \in \mathbb{N}} : n \in \mathbb{N}\} \) the positions at time \( n \) of the independent random walkers (labeled by \( k \)) in a volume of linear size \( L \), then the process \( \{R_t = (R_{k,t})_{k \in \mathbb{N}} : t \in \mathbb{R}^+ \} \) defined by \( R_{k,t} := \lim_{L \to \infty} X^{(L)}_k(t\omega) \) will be a family of anisotropic independent Brownian motions with drift, satisfying the stochastic differential equation \( dR_{k,t} = 2AEdt + \sqrt{A}dB_{k,t} \) with \( A \) the \( d \times d \) diagonal matrix with elements \( A_{ss} = 2a_s \) and \( B_{k,t} \in \mathbb{R}^d \) denoting the standard Brownian motion. An immediate computation gives
\[
\lim_{L \to \infty} \frac{\mu_{RW}^L(\lambda)}{L^{d-2}} = \mu_{BM}^L(\lambda) = \rho \sum_{s=1}^{d} a_s \lambda_s (\lambda_s + 2E_s) \tag{10}
\]
where \( \mu_{BM}^L(\lambda) := \lim_{L \to \infty} \frac{1}{L} \ln \left[ \mathbb{E}(e^{L^{-1}\lambda R_t}) \right] \). From the explicit expression (10) we see that a spatial fluctuation relation holds, i.e.,
\[
\mu_{BM}(\lambda) = \mu_{BM}(U^{-1}) \tag{11}
\]
with \( U \) such that \( UAU^t = A \). For \( d = 2 \), \( U \) takes the form given at the end of the caption of Fig. 1. As we shall see below, such a relation can be traced back to the invariance of the path measure of the anisotropic Brownian motion under a spatial transformation \( U \) such that \( UAU^t = A \) (in agreement with the findings of [16,17] in the context of the Macroscopic Fluctuation Theory [3]).

**Proposition.** Consider a probability measure on path space \( \Omega \) of the form
\[
P(d\omega) = e^{H(\omega)} P_0(d\omega) \tag{13}
\]
where \( P_0 \) is \( \mathcal{R} \)-invariant. For all \( \varphi : \Omega \to \mathbb{R} \) we have the identity
\[
\mathbb{E}(e^{\varphi(\omega)}) = \mathbb{E}(e^{\varphi(\omega) U \omega^{-1} H}) \tag{14}
\]
consider \( \Omega = C([0, T], \mathbb{R}^d) \), i.e. the set of all continuos paths up to time \( T \) taking value in \( \mathbb{R}^d \), and
\[
\mathcal{R}(\omega)_t = U \omega_t \tag{12}
\]
with \( U : \mathbb{R}^d \to \mathbb{R}^d \) an invertible spatial transformation.
Proof.

\[
\int e^{\varphi \circ H} dP = \int e^{\varphi} e^{H \circ \varphi^{-1}} dP_0 = \int e^{\varphi} e^{H \circ \varphi^{-1} - H} dP
\]

In the particular case that \( \varphi = -\gamma(H \circ \varphi^{-1} - H) \) with \( \gamma \in \mathbb{R} \), the relation \( (14) \) gives \( \mathbb{E}(e^{(H \circ \varphi^{-1})}) = \mathbb{E}\left(e^{(1-\gamma)(H \circ \varphi^{-1} - H)}\right) \). More specifically, for a transformation such that \( \varphi = \varphi^{-1} \), this identity is exactly a symmetry of the form of the standard fluctuation theorem for the quantity \( H \circ \varphi^{-1} - H \).

Denoting by \( P \circ \varphi \) the image measure of \( P \) under \( \varphi \), i.e., \( \int fd(P \circ \varphi) = \int(f \circ \varphi)dP \), it can be readily verified that \( dP = e^H dP_0 \) implies that \( d(P \circ \varphi) = e^{H \circ \varphi^{-1}} dP_0 \).

Therefore, as we shall use below, we can write

\[
e^{H \circ \varphi^{-1} - H} = \frac{d(P \circ \varphi)}{dP} = \frac{d(P \circ \varphi)}{dP_0} \cdot \frac{dP_0}{dP}. \tag{15}
\]

The abstract setting of the proposition above can be used to derive the spatial fluctuation theorem for finite time \( T \) in the context of interacting diffusions. We shall illustrate this by considering the overdamped Langevin dynamics \( \{X_t = (X_{k,t})_{k \in \mathbb{N}} : t \in \mathbb{R}^+\} \) describing \( N \) particles (labeled by \( k \)) subject to a drift vector which can arise from an applied force to each particle \( F \) and/or a conservative potential \( V \), with a positive definite constant diffusion matrix \( A \). Then the stochastic differential equation for the \( k \)th particle reads

\[
dX_{k,t} = F(X_{k,t}) dt + \nabla_k V(X_t) dt + \sqrt{A} dB_{k,t}, \tag{16}
\]

where \( B_{k,t} \) is again a standard Brownian motion. Notice that \( V \) can model a self-potential as well as an interaction potential. To obtain the new process applying a transformation \( \varphi \) of the type \( (12) \), we remind that if \( X \) is multivariate normal distributed with mean zero and covariance matrix \( A \), then for any \( d \times d \) matrix \( U \), \( UX \) is multivariate normal distributed with mean zero and covariance \( UAU^t \). As a consequence if \( Y_{k,t} = UX_{k,t} \) then

\[
dY_{k,t} = U(F(−Y_{k,t}) dt + U \nabla_k V(Y_t) dt + \sqrt{AU} dB_{k,t}
\]

where we denote \( U^{-1} Y_t \) the collection \( U^{-1} Y_{k,t} \) \( \forall k \). For the process \( Y_t \) to be absolutely continuous w.r.t. the \( X_t \) process, we need that \( UAU^t = A \). Moreover, assuming that the potential is invariant under the transformation \( U \), i.e. \( V(Ux) = V(x) \), the process \( Y_{k,t} \) satisfies

\[
dY_{k,t} = U(F(U^{-1} Y_{k,t}) dt + \nabla_k V(Y_t) dt + \sqrt{A} dB_{k,t}. \tag{17}
\]

Thus, the process whose paths’ distribution is invariant under the transformation \( \varphi \) is

\[
dZ_{k,t} = \nabla_k V(Z_t) dt + \sqrt{A} dB_{k,t}. \tag{18}
\]

By using the Girsanov formula \( \text{[21]} \) one can compute \( dP_X = e^H dP_Z \), i.e. the relative density between the path space measure \( P_X \) of the process \( \{X_t = (X_{k,t})_{k \in \mathbb{N}} : t \in \mathbb{R}^+\} \) and the path space measure \( P_Z \) of the process \( \{Z_t = (Z_{k,t})_{k \in \mathbb{N}} : t \in \mathbb{R}^+\} \). Analogously the measure \( P_Y \) of the process \( \{Y_t = (Y_{k,t})_{k \in \mathbb{N}} : t \in \mathbb{R}^+\} \) and the measure \( P_Z \) are related by \( d(P_X \circ \varphi) = dP_Y = e^{H \circ \varphi^{-1}} dP_Z \). Hence, if we denote \( U \) and \( \tilde{A} \) as the \( N \times d \) block diagonal matrices with elements \( U \) and \( A \) respectively on the diagonal and \( F \) as the \( N \times d \) column vector consisting in copying \( N \) times the vector \( F \), then by applying \( (15) \) we get

\[
(\circ \varphi^{-1} - H)(\omega) = \int_0^T \tilde{A}^{-1}(\tilde{U} \tilde{F}(\tilde{U}^{-1} \omega_t) - \tilde{F}(\omega_t)) d\omega_t - \frac{1}{2} \int_0^T (\tilde{U} \tilde{F}(\tilde{U}^{-1} \omega_t) \cdot \tilde{A}^{-1}(\tilde{U} \tilde{F}(\tilde{U}^{-1} \omega_t) dt + \frac{1}{2} \int_0^T \tilde{F}(\omega_t) \cdot \tilde{A}^{-1} \tilde{F}(\omega_t) dt. \tag{19}
\]

In addition, if the force \( F \) is constant we find (we put \( \omega_0 = 0 \))

\[
(\circ \varphi^{-1} - H)(\omega) = A^{-1}(UF - F) \cdot Q_T(\omega) \tag{20}
\]

with

\[
Q_T(\omega) = \sum_{k=1}^{N} (\omega_{k,T} - \int_0^T \nabla_k V(\omega_t) dt), \tag{21}
\]

Furthermore, by choosing \( \varphi(\omega) = \lambda \cdot Q_T(\omega) \), with \( \lambda \in \mathbb{R}^d \), we have that \( (\circ \varphi)(\omega) = \lambda \cdot UQ_T(\omega) = U^t \lambda \cdot Q_T(\omega) \) and from \( (14) \) and \( (20) \) we get \( \mathbb{E}(e^{\varphi U^t \lambda \cdot Q_T}) = \mathbb{E}(e^{\lambda A^{-1}(UF - F) \cdot Q_T}) \). By defining \( Z_T(\lambda) := \mathbb{E}(e^{\lambda^t \cdot Z_T}) \) it readily follows that

\[
Z_T(\lambda) = Z_T(\lambda^t)(\lambda + A^{-1} F - A^{-1} F)^t \tag{22}
\]

which is the analogous relation to the previously found for the discrete setting \( [9] \). Notice that if \( A \) is the diagonal matrix with elements \( A_{k,k} = 2a_{k,k} \), \( F = AF \), \( V = 0 \) and \( \mu_B(\lambda) := \lim_{T \rightarrow \infty} T^{-1} \ln Z_T(\lambda) \) we recover from the above equation the previous result \( (11) \) for the anisotropic Brownian motion with drift.

Conclusions: In this letter we have derived a spatial fluctuation theorem (SFT) for interacting particle systems and interacting diffusions driven out of equilibrium by an external field. It has been proved in both cases that the SFT can be traced back to an invariance property of the microscopic path space measure under spatial transformations. Remarkably, this result holds for finite time in the case of interacting diffusions \( [22] \), with \( UAU^t = A \) and \( V(Ux) = V(x) \), as well as for finite volume for interacting particle systems \( [6] \). In the latter, the spatial transformations yielding the SFT correspond to discrete symmetries associated with the underlying lattice geometry of the system at hand. However, as the system linear
size increases these transformations become continuous, as is depicted in Fig. [1] for the particular case of non-interacting random walkers.

The SFT gives a new perspective on how the microscopic symmetries of the system are reflected at the fluctuating level. Whereas the standard fluctuation theorem is based on time reversal-symmetry of the microscopic dynamics, the SFT proves that the spatial symmetries have also a word to say. It is worth emphasizing that from the SFT new hierarchies for the currents cumulants and for the non-linear response can be obtained [16, 17]. In addition, from an experimental point of view (see e.g. [22, 23]), checking the SFT and its consequences would also be very interesting and statistically feasible since one does not have necessarily to measure the unlikely inverse fluctuation of the vectorial observable at hand, but just a spatial transformation of it. The study of the full Langevin dynamics (including inertia) also remains an open problem.

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