

Energy-exchange stochastic models for non-equilibrium

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Non-equilibrium steady states are subject to intense investigations but still poorly understood. For instance, the derivation of Fourier law in Hamiltonian systems is a problem that still poses several obstacles. In order to investigate non-equilibrium systems, stochastic models of energy-exchange have been introduced and they have been used to identify universal properties of non-equilibrium. In these notes, after a brief review of the problem of anomalous transport in 1-dimensional Hamiltonian systems, some boundary-driven interacting random systems are considered and the “duality approach” to their rigorous mathematical treatment is reviewed. Duality theory, of which a brief introduction is given, is a powerful technique to deal with Markov processes and interacting particle systems. The content of these notes is mainly based on the papers [10, 11, 12].

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1 Fourier's law and anomalous transport

The models we would like to discuss are motivated by two fundamental open problems in mathematical statistical physics:

1. deriving phenomenological laws of non-equilibrium statistical physics for a microscopic interacting model;
2. understanding the general structure and properties of the probability measure describing a system in a non-equilibrium steady state.

In this opening section we briefly discuss the first problem, while we defer the second to the study of the specific models discussed below. We focus on the Fourier's law

$$\langle J \rangle = \kappa \cdot \nabla T \quad (1)$$

describing the heat flow across a metal bar when we heat the bar from one side and cool it from the other. The law bears its name to J.B.J. Fourier [8] who discovered it in 1822. In (1) the left hand side $\langle J \rangle$ is the average energy current per unit time and per unit of surface, whereas in the right side κ is the material's conductivity and ∇T is the temperature gradient. One would like to derive this phenomenological law (and the linear heat equation describing the diffusive heat spreading) from a simplified, yet realistic, mathematical model. The simplest setting is obtained by considering a one-dimensional system with L sites coupled to two external reservoirs imposing temperatures T_l and T_r at the extremes. A crucial distinction emerges as those two parameters are varied:

- If $T_l = T_r$ then in the long time limit the system reaches an equilibrium state described by the Boltzmann-Gibbs probability distribution.
- If $T_l \neq T_r$ then a non-equilibrium state arises. We will be interested in those cases in which a stationary measure sets in the long time limit. Such invariant measure will be called in the sequel the non-equilibrium probability measure.

1.1 Hamiltonian models

In the basic setting described above one considers a bulk part and a boundary contribution. If one starts from the assumption that the micro-world evolution is described by Newton equations, then the bulk part of the model consists of particles whose dynamics is encoded in the Hamiltonian

$$H_L(q, p) = \sum_{i=1}^L \left(\frac{p_i^2}{2} + U(q_i) \right) + \sum_{i=1}^{L-1} V(q_{i+1} - q_i) \quad (2)$$

Here the particles are assumed to have unit mass, position and momentum coordinates are denoted by $(q, p) = (q_1, \dots, q_L, p_1, \dots, p_L)$. The Hamiltonian has a local contribution (including kinetic energy and a site potential U) and a nearest-neighbor interaction potential V . For the boundaries, one option is to model the reservoirs by adding to the velocities of the first and the last particles an Ornstein-Uhlenbeck process that fixes the temperatures via a fluctuation-dissipation mechanism¹. Therefore the full equations of motion read

$$\begin{cases} dq_i = p_i dt \\ dp_i = -\frac{\partial H_L}{\partial q_i} dt + \delta_{i,1}(-\gamma p_1 + \sqrt{2\gamma T_l} dW^{(l)}) + \delta_{i,L}(-\gamma p_L + \sqrt{2\gamma T_r} dW^{(r)}) \end{cases}$$

¹It is also possible to work with deterministic thermostats (e.g. Nosé-Hoover or isokinetic), however this setting will not be discussed here (see [17] for more on this)

where $W^{(l)}$ and $W^{(r)}$ are two independent standard Brownian motion and $0 < \gamma < \infty$ is a parameter tuning the coupling to reservoirs. The microscopic definition of the observables appearing in the Fourier law (1) follows from the discretization of the continuity equation

$$\frac{\partial \rho_i}{\partial t} = -(j_i - j_{i-1})$$

where ρ_i denotes the energy density at site i and j_i is the energy current across the bound $(i, i + 1)$. As discussed in [17], this leads to the definition of the current in the bulk, i.e. for $i \in \{1, \dots, L - 1\}$

$$j_i := -\frac{1}{2}(p_i + p_{i+1}) \frac{\partial H_L}{\partial q_i} \quad (3)$$

and the average current for a system of size L is given by

$$\langle J^{(L)} \rangle := \frac{1}{L-1} \sum_{i=1}^{L-1} \langle j_i \rangle \quad (4)$$

where $\langle \cdot \rangle$ denotes expectation with respect to the stationary non-equilibrium probability measure. The details of the computation leading to the definition in (3) and (4) can be found in [17]. As for the temperature the standard definition is given by twice the average kinetic energy ², yielding

$$T_i := \langle p_i^2 \rangle \quad (5)$$

Combining together (4) and (5) and assuming validity of Fourier's law (1) with linear stationary energy profiles, one obtains a definition of the conductivity for a system of size L

$$\kappa_L := \langle J^{(L)} \rangle \frac{L}{T_l - T_r}$$

and considering the thermodynamic limit one has the definition of the system conductivity

$$\kappa := \lim_{L \rightarrow \infty} \kappa_L .$$

1.2 Stylized properties of Fourier's law in 1d Hamiltonian models

With reference to the generic Hamiltonian (2), the following picture emerges from numerical and analytical studies of several models.

- i) For harmonic oscillators with $U(x) = 0$ and $V(x) = \frac{1}{2}x^2$ Lebowitz, Lieb and Rieder [19] proved that $\kappa_L \simeq L$. The result is rooted in the fact that the L degrees of freedom can be decoupled into normal modes, which transport ballistically the heat from one side to the other. Furthermore [19] proves also that the non-equilibrium invariant measure is a multivariate Gaussian measure.
- ii) For non-linear oscillator chains with a non-vanishing on site potential (i.e. $U(x) \neq 0$) a finite conductivity is found [17]. The reason is that the on-site potential acts as a source of scattering among the normal modes. However this case is believed to be quite unrealistic.

²Other definitions of the temperature are possible, cfr [20, 13].

- iii) For non-linear oscillator chains with translation invariant interaction (i.e. $U(x) = 0$) one typically finds $\kappa_L \sim L^\alpha$ with $0 < \alpha < 1$. This is the phenomenon of *anomalous transport*, whose origin has been linked to the presence of additional conserved quantities (momentum, besides energy) for the bulk dynamics. The result is supported both by numerical analysis of several models (a much studied case is the FPU- β model with a quartic potential $V(x) = \frac{1}{2}x^2 + \frac{1}{4}\beta x^4$) as well as by analytical studies based on mode-coupling theory [17] and nonlinear fluctuating hydrodynamics [1, 21]. From the numerical experiments the value of the exponent $\alpha \in [0.3, 0.5]$.
- iv) An exceptional case is given by the rotor model with potential $U(x) = 0$, $V(x) = 1 - \cos(x)$. In this case, despite conservation of momentum, the finite volume conductivity scales as $k_L = a + \frac{b}{L}$, yielding a finite conductivity in the thermodynamic limit. This was found by two (independent) numerical studies of the works [14, 9].
- v) Recently it has been claimed [22] that, despite the divergence observed in numerical simulations of finite system sizes, non-linear oscillator chains with $U(x) = 0$ and asymmetric potential $V(x) \neq -V(x)$ have finite asymptotic thermal conductivity at low temperatures and anomalous transport at high temperatures. The claim has been contradicted in [6].

1.3 Stochastic models

The use of stochastic models to model the bulk system is a further simplifying assumption. In this approach the Hamiltonian dynamics is replaced by a stochastic evolution and exact solutions can be obtained. The first model of this type was introduced by Kipnis, Marchioro and Presutti [16] in 1982. They considered a model in which the energy is uniformly redistributed among nearest neighbor particles, thus providing an efficient mechanism of energy transport across the extended system. They proved the validity of Fourier law and introduced the duality approach that is the core of these lectures. More recent works include [2], where the case of harmonic oscillators with an energy conserving stochastic noise has been studied. Remarkably they find that if the noise is only energy-conserving then the conductivity is finite. On the contrary, if the noise conserves both energy and momentum then $\kappa_L \sim \sqrt{L}$, thus strengthening the claim that conservation of momentum leads (in general) to anomalous transport.

1.4 From Hamiltonian to stochastic

We conclude this introduction by considering a model that has been introduced in [10], it serves as a minimalistic model to go from Hamiltonian to stochastic dynamics, in the sense that in the high-energy limit the deterministic dynamics is well-approximated by a stochastic dynamics. Consider the Hamiltonian

$$H_L(q, p) = \sum_{i=1}^L \frac{(p_i - A_i(q))^2}{2} \quad (6)$$

where $A(q) = (A_1(q), \dots, A_L(q))$ is a generalized "vector" potential in \mathbb{R}^L . In this model there is a non-trivial distinction between momentum and velocities. The equations of motion read

$$\begin{cases} \frac{dq_i}{dt} = v_i \\ \frac{dv_i}{dt} = \sum_j B_{ij} v_j \end{cases}$$

where the “magnetic fields” are obtained from the generalized potential as

$$B_{i,j}(q) = \frac{\partial A_i(q)}{\partial q_j} - \frac{\partial A_j(q)}{\partial q_i}$$

Since the fields form an antisymmetric matrix, i.e. $B_{i,j} = -B_{j,i}$, the total energy is conserved:

$$\frac{d}{dt} \sum_i \frac{v_i^2}{2} = \sum_i v_i \frac{dv_i}{dt} = \sum_{i,j} B_{ij} v_i v_j = 0.$$

The model (6), coupled to reservoirs, has been studied using numerical simulations in [10] for several choices of the vector potential and the thermal conductivity measured for different system sizes and different temperature. From those studies it has been found that the system has always a finite conductivity. Moreover, as the temperature is increased, the conductivity approaches an asymptotic constant value (see figure 1).

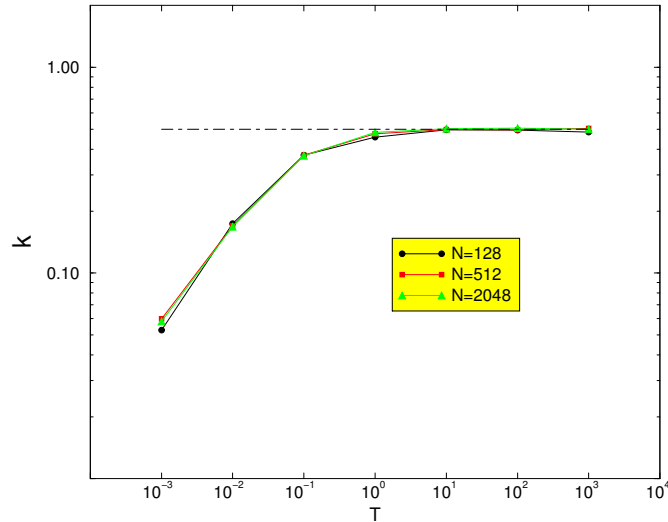


Figure 1: Heat conductivity as a function of the temperature for the model (6) with reservoirs and for a particular choice of the vector potential A . Picture taken from [10].

The findings of the numerical study of the model suggested that in the high-energy limit the deterministic dynamics could be substituted with a random evolution. The stochastic model that will be discussed in these lectures is indeed obtained by replacing the deterministic magnetic fields $B_{i,j}$ with a family of random fields given by independent Brownian motions [10].

2 The Brownian Momentum Process (BMP)

Definition 2.1 (BMP, stochastic differential equations) For a graph $G_L = (V_L, E_L)$ where V_L is the vertex set of cardinality $L \in \mathbb{N}$ and E_L is the edges set, the Brownian Momentum Process $(X(t))_{t \geq 0} = (X_i(t))_{i \in V_L, t \geq 0}$ is a diffusion process that takes value in \mathbb{R}^L and satisfies the SDE (in Stratonovich sense)

$$dX_i = \sum_{j: (i,j) \in E_L} X_j \circ dW_{ij}(t)$$

where for $1 \leq i < j \leq L$ the $W_{ij}(t)$ are a family of independent standard Brownian motion and for $1 \geq i > j \geq L$ we set $W_{ij}(t) = -W_{ji}(t)$. Moreover $W_{ii}(t) = 0$ for $i \in V_L$.

Remark 2.2 The $X(t)$ vector collects the velocities of L particles at time $t > 0$. For any graph G_L it follows from the definition above that the total kinetic energy is conserved

$$\frac{d}{dt} \left(\frac{1}{2} \sum_{i \in V_L} X_i^2(t) \right) = 0$$

An alternative definitions of a continuous time Markov process is obtained by specifying its generator.

Definition 2.3 (BMP, generator) The generator of the BMP process on the graph $G_L = (V_L, E_L)$ is the second order differential operator \mathcal{L}^{BMP} defined on smooth functions $f : \mathbb{R}^L \rightarrow \mathbb{R}$ as

$$\mathcal{L}^{BMP} f(x) = \sum_{(i,j) \in E_L} \mathcal{L}_{ij}^{BMP} f(x) = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x) \quad (7)$$

Remark 2.4 We remind that the generator \mathcal{L} of a Markov process gives the infinitesimal evolution of the expectations of observables. Namely, for a measurable function f

$$\mathcal{L}f(x) = \lim_{h \rightarrow 0} \frac{\mathbb{E}_x [f(X_h)] - f(x)}{h} = \frac{d}{dt} \mathbb{E}_x [f(X_t)] \Big|_{t=0}$$

where $\mathbb{E}_x [f(X_t)] = \mathbb{E} [f(X_t) | X_0 = x]$. As a consequence

$$\frac{d}{dt} \mathbb{E} [f(X_t)] = \mathbb{E} [\mathcal{L}f(X_t)] .$$

By exponentiating the generator one (formally) obtains the semigroup $S_t = \exp[t\mathcal{L}]$. Namely, assuming the Hille-Yoshida theorem [18] can be applied, one has

$$\mathbb{E}_x [f(X_t)] = S_t f(x).$$

The last characterization of a Markov stochastic process is obtained by the forward Kolmogorov equation (also called Fokker-Planck equation in the context of diffusions). For the forward Kolmogorov equation to exist it is required that the process is absolutely continuous with respect to the Lebesgue measure.

Definition 2.5 (BMP, Fokker-Planck equation) The probability density function $p(x, t) : \mathbb{R}^L \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of the BMP process at time $t > 0$ on the graph $G_L = (V_L, E_L)$ started from the density g at time $t = 0$ is given by

$$\begin{cases} \frac{d}{dt} p(x, t) = ((\mathcal{L}^{BEP})^* p)(x, t) \\ p(x, 0) = g(x) \end{cases} \quad (8)$$

where $(\mathcal{L}^{BEP})^*$ is the adjoint in $L^2(dx)$ of \mathcal{L}^{BEP} in (7). It turns out that $(\mathcal{L}^{BEP})^* = \mathcal{L}^{BEP}$.

Remark 2.6 From the Fokker-Planck equation it is easy to check that product measure with marginal given by $\mathcal{N}(0, T)$, i.e. centered Gaussian with variance T , are invariant. It is believed that the family of such product measure, labeled by the variance $T > 0$, exhaust all the ergodic measures [15]. In other words it should be possible to write any other invariant measure as a convex combination of those product measures.

2.1 Brownian Momentum Process with reservoirs

In order to make connection with the heat transport discussed in the first section we specify to a graph G_L , which is given by the one-dimensional lattice of L sites with edges between nearest neighbor vertices. Moreover we add to the first and last site two reservoirs at temperatures T_l and T_r modeled as Ornstein-Uhlenbeck processes.

Definition 2.7 (BMP with reservoirs) *The generator of the BMP process with reservoirs is*

$$\mathcal{L}^{BMP, res} = \sum_{i=1}^{L-1} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 - x_1 \frac{\partial}{\partial x_1} + T_l \frac{\partial^2}{\partial x_1^2} - x_L \frac{\partial}{\partial x_L} + T_r \frac{\partial^2}{\partial x_L^2} \quad (9)$$

If $T_l = T_r = T$ then a unique stationary measure is selected, namely the product of centered Gaussians with variance T with probability density function

$$p(x) = \prod_{i=1}^L \frac{1}{\sqrt{2\pi T}} e^{-\frac{x_i^2}{2T}}.$$

The question we would like to address is what can be said about the situation in which $T_l \neq T_r$. We will see that a characterization of the non-equilibrium invariant measure can be achieved by using stochastic duality.

3 Duality theory

Duality theory is a powerful technique to deal with stochastic processes. In a nutshell, the main idea behind duality is to study a given process making use of a simpler dual one. In particular the connection between the two processes occurs on a set of so-called duality functions.

3.1 Duality

Definition 3.1 (Duality) *Let $X = (X(t))_{t \geq 0}$ and $Y = (Y(t))_{t \geq 0}$ be two Markov processes with state spaces Ω , respectively Ω^{dual} , and generator \mathcal{L} , respectively \mathcal{L}^{dual} . We say that X is dual to Y with duality function $D : \Omega \times \Omega^{dual} \mapsto \mathbb{R}$ if*

$$\mathbb{E}_x[D(X(t), y)] = \mathbb{E}_y[D(x, Y(t))] , \quad (10)$$

for all $(x, y) \in \Omega \times \Omega^{dual}$ and $t \geq 0$.

Remark 3.2 *Markov processes X and Y are dual on function D iff*

$$[\mathcal{L}D(\cdot, y)](x) = [\mathcal{L}^{dual}D(x, \cdot)](y) \quad (11)$$

Indeed, informally, we have the following series of equalities

$$\mathbb{E}_x[D(X_t, y)] = (e^{t\mathcal{L}}D(\cdot, y))(x) = (e^{t\mathcal{L}^{dual}}D(x, \cdot))(y) = \mathbb{E}_y[D(x, Y_t)]$$

3.2 Algebraic approach to duality theory

How to find a dual process? How to find duality-function? The following scheme has been put forward in a series of works [11, 12]

- Duality arises as a change of representation of an abstract operator that belongs to a Lie-algebra.
- Duality functions correspond to the intertwiners between the two representations.

We shall illustrate this approach by considering the case of Brownian Momentum Process and its underlying $SU(1, 1)$ Lie algebra structure. The existence of a dual process often allows to simplify the analysis of the process at hand. In the context of boundary driven non-equilibrium systems, further simplifications take place. A list of consequences of duality theory includes:

1. *From continuous to discrete:* interacting diffusions can be studied via interacting particles systems.
2. *From reservoirs to absorbing boundaries:* stationary state of boundary-driven processes with reservoirs can be fully characterized by dual processes with absorbing boundaries.
3. *From many to few:* n -point correlation functions of a system of site L can be studied using n dual walkers.

4 The Symmetric Inclusion Process (SIP)

Definition 4.1 (SIP(m), generator) *Given the graph $G_L = (E_L, V_L)$ the Symmetric Inclusion Process with parameter $m \in \mathbb{R}_+$ is the continuous time Markov chain $(\eta(t))_{t>0} = (\eta_i(t))_{i \in V_L, t \geq 0}$ with state space \mathbb{N}^L with generator*

$$\begin{aligned} \mathcal{L}^{SIP(m)} f(\eta) &= \sum_{(i,j) \in E_L} \mathcal{L}_{i,j}^{SIP(m)} f(\eta) = \\ & \sum_{(i,j) \in E_L} \eta_i \left(\frac{m}{2} + \eta_j \right) [f(\eta^{i,j}) - f(\eta)] + \eta_j \left(\frac{m}{2} + \eta_i \right) [f(\eta^{j,i}) - f(\eta)] \end{aligned}$$

where the vector $\eta^{i,j}$ is obtained from the configuration η by moving one particle from site i to site j , i.e.

$$\eta^{i,j} = (\eta_1, \dots, \eta_i - 1, \dots, \eta_j + 1, \dots, \eta_L)$$

Remark 4.2 *The SIP process is an interacting particles system with attractive interactions. The stationary reversible measures are product with marginals Neg Bin $\left(\frac{m}{2}, p\right)$, i.e.*

$$\mathbb{P}(\eta_1 = n_1, \dots, \eta_V = n_V) = \prod_{i \in V} p^{n_i} (1-p)^{\frac{m}{2}} \binom{n_i + \frac{m}{2} - 1}{\frac{m}{2} - 1}$$

We remind that Neg Bin $\left(\frac{m}{2}, p\right)$ represents the number of trials before the $\left(\frac{m}{2}\right)^{th}$ failure in a sequence of independent Bernoulli trials with success probability p .

4.1 SIP with absorbing boundaries

Definition 4.3 (SIP(1) with absorbing boundaries) *We add two extra sites, i.e. the configurations are $\eta = (\eta_0, \eta_1, \dots, \eta_L, \eta_{L+1})$. The generator of Symmetric Inclusion Process with absorbing boundaries and parameter $m = 1$ is*

$$\begin{aligned} \mathcal{L}^{SIP(1),abs} f(\eta) = & \sum_{i=1}^{L-1} \eta_i \left(\frac{1}{2} + \eta_{i+1} \right) [f(\eta^{i,i+1}) - f(\eta)] + \\ & \eta_{i+1} \left(\frac{1}{2} + \eta_i \right) [f(\eta^{i+1,i}) - f(\eta)] + \\ & \frac{\eta_1}{2} [f(\eta^{1,0}) - f(\eta)] + \frac{\eta_L}{2} [f(\eta^{L,L+1}) - f(\eta)] \end{aligned}$$

4.2 Duality between BMP with reservoirs and SIP(1) with absorbing boundaries

The main result, which allows studying the non-equilibrium invariant measure of the Brownian Momentum Process, is contained in the following theorem.

Theorem 4.4 (Duality BMP/SIP(1)) *The BMP with reservoirs is dual to the SIP(1) with absorbing boundaries on duality function*

$$D(x, \eta) = T_l^{\eta_0} \left(\prod_{i=1}^L \frac{x_i^{2\eta_i}}{(2\eta_i - 1)!!} \right) T_r^{\eta_{L+1}}$$

Proof. From equation (11) the proof is a consequence of the identity

$$(\mathcal{L}^{BMP,res} D(\cdot, \eta))(x) = (\mathcal{L}^{SIP(1),abs} D(x, \cdot))(\eta)$$

The above equation can be verified by a direct explicit computation. □

5 Intermezzo: SU(1,1) algebra

We recall that if A and B are operators working on a common domain, the commutator of A and B is $[A, B] := AB - BA$.

Definition 5.1 (SU(1,1) algebra) *For a graph $G_L = (V_L, E_L)$, we define the $SU(1,1)$ algebra as the algebra of elements generated by $\{K_i^+, K_i^-, K_i^0\}_{i \in V_L}$ satisfying the commutation relations*

$$[K_i^-, K_j^+] = 2K_i^0 \delta_{ij}, \quad [K_i^0, K_j^\pm] = \pm K_i^\pm \delta_{ij} \quad (12)$$

The duality between BMP and SIP process can be seen as a change of representation of the abstract operator \mathbf{L} that is a linear combination of the generators of the $SU(1,1)$ algebra

$$\begin{aligned} \mathbf{L} = & \sum_{i=1}^{L-1} \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^0 K_{i+1}^0 + \frac{1}{8} \right) + \\ & - \left(2K_1^0 + \frac{1}{2} + T_l 2K_1^- \right) - \left(2K_L^0 + \frac{1}{2} + T_r 2K_1^- \right) \end{aligned}$$

There exists a representation in terms of differential operators where the generators of the $SU(1,1)$ algebra are given by

$$K_i^+ = \frac{1}{2}x_i^2, \quad K_i^- = \frac{1}{2}\partial_i^2, \quad K_i^0 = \frac{1}{4}(x_i\partial_i + \partial_ix_i).$$

In this case one finds that $\mathbf{L} = \mathcal{L}^{BMP,res}$. Another discrete representation is obtained in terms of infinite dimensional matrices by writing

$$\mathcal{K}_i^+|\eta_i\rangle = (\eta_i + \frac{1}{2})|\eta_i + 1\rangle \quad \mathcal{K}_i^-|\eta_i\rangle = \eta_i|\eta_i - 1\rangle \quad \mathcal{K}_i^0|\eta_i\rangle = (\eta_i + \frac{1}{4})|\eta_i\rangle$$

where $|\eta_i\rangle$ denotes the vector with all components equal to 0 except the i^{th} component equal to 1. In this case one finds that $\mathbf{L} = \mathcal{L}^{SIP(1),abs}$. Duality functions are the intertwiner between the two representations and they are obtained by imposing relation (11) for all the algebra generators. In particular,

$$K_i^+d(\cdot, \eta_i)(x_i) = \mathcal{K}_i^+d(x_i, \cdot)(\eta_i)$$

leads straight to

$$d(x_i, \eta_i) = \frac{x_i^{2\eta_i}}{(2\eta_i - 1)!!}.$$

6 Correlation functions in the stationary state

Next theorem shows that the moments of energy (i.e. square of the velocity) of the BMP process with reservoirs can be characterize via duality.

Theorem 6.1 (Moments of BMP) *Let η be a configuration of the SIP(1) process with absorbing boundaries and denote by $|\eta| = \sum_{i=1}^L \eta_i$ the total number of SIP dual walkers. Let*

$$p_\eta(a, b) = \mathbb{P}(\eta_0(\infty) = a, \eta_{L+1}(\infty) = b \mid \eta(0) = \eta). \quad (13)$$

be the probability that a SIP walkers are absorbed at site 0 and b of them are absorbed at site L + 1. Then denoting by $\langle \cdot \rangle_L$ the expectation in the non- equilibrium stationary state of the BMP process with reservoirs one has

$$\langle D(X, \eta) \rangle_L = \sum_{a, b : a+b=|\eta|} T_l^a T_r^b p_\eta(a, b) \quad (14)$$

Proof.

Consider the BMP process started from the initial measure ν . Then

$$\begin{aligned} \langle D(X, \eta) \rangle_L &= \lim_{t \rightarrow \infty} \int \mathbb{E}_x [D(X(t), \eta)] \nu(dx) \\ &= \lim_{t \rightarrow \infty} \int \mathbb{E}_\eta [D(x, \eta(t))] \nu(dx) \\ &= \int \lim_{t \rightarrow \infty} \mathbb{E}_\eta [D(x, \eta(t))] \nu(dx) \\ &= \int \nu(dx) \mathbb{E}_\eta (T_l^{\eta_0(\infty)} T_r^{\eta_{L+1}(\infty)}) \\ &= \sum_{a, b : a+b=|\eta|} p_\eta(a, b) T_l^a T_r^b \end{aligned}$$

□

Example 1: temperature profile

In particular it follows that if $\eta = (0, \dots, 0, 1, 0, \dots, 0)$ is a null vector with 1 in the i^{th} position, then $D(X, \eta) = X_i^2$. Denoting by $(R(t))_{t>0}$ a continuous time symmetric random walker jumping at rate 1/2 and absorbed at the boundaries $\{0, L+1\}$ also with rate 1/2 one has

$$\begin{aligned} \langle X_i^2 \rangle &= T_l p_\eta(1, 0) + T_r p_\eta(0, 1) \\ &= T_l \mathbb{P}(R(\infty) = 0 | R(0) = i) + T_r \mathbb{P}(R(\infty) = L+1 | R(0) = i) \\ &= T_l \left(1 - \frac{i}{L+1}\right) + T_r \left(\frac{i}{L+1}\right) \\ &= T_l + \frac{T_r - T_l}{L+1} i \end{aligned}$$

Example 2: energy covariance

If $\eta = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ is a null vector with 1 in the i^{th} position and in the j^{th} position, then $D(X, \eta) = X_i^2 X_j^2$. Therefore

$$\begin{aligned} \langle X_i^2 X_j^2 \rangle &= T_l^2 p_\eta(2, 0) + T_r^2 p_\eta(0, 2) + T_l T_r (p_\eta(1, 0) + p_\eta(0, 1)) \\ &= T_l^2 \mathbb{P}_{ij}(R_1(\infty) = R_2(\infty) = 0) + T_r^2 \mathbb{P}_{ij}(R_1(\infty) = R_2(\infty) = L+1) \\ &\quad + T_l T_r [\mathbb{P}_{ij}(R_1(\infty) = 0, R_2(\infty) = L+1) + \mathbb{P}_{ij}(R_1(\infty) = L+1, R_2(\infty) = 0)] \end{aligned}$$

where $(R_1(t), R_2(t))_{t>0}$ are two continuous time SIP walkers absorbed at the boundaries $\{0, L+1\}$. A computation gives [11]

$$\langle X_i^2 X_j^2 \rangle - \langle X_i^2 \rangle \langle X_j^2 \rangle = \frac{2i(L+1-j)}{(L+3)(L+1)^2} (T_l - T_r)^2$$

7 Redistribution model

The example of duality between BMP and SIP(1) process can be generalized in several ways. For instance one can define energy redistribution jump process by considering instantaneous thermalization limit of the BMP process.

7.1 The Brownian Energy Process BEP(m)

It is convenient to start from a ladder-graph with m copies of the one dimensional lattice of L sites. The Brownian Momentum Process on such graph has generator

$$\mathcal{L}^{BMP(m)} = \sum_{i=1}^{L-1} \sum_{\alpha, \beta=1}^m (x_{i,\alpha} \partial_{i+1,\beta} - x_{i+1,\beta} \partial_{i,\alpha})^2 \quad (15)$$

Let $Z_i(t) = \sum_{\alpha=1}^m X_{i,\alpha}^2(t)$ be the energy at site i at time $t \geq 0$. Then one can check that $(Z(t))_{t \geq 0} = (Z_i(t))_{i \in V_L, t \geq 0}$ defines a Markov process called the Brownian Energy Process with parameter $m \in \mathbb{R}$ (BEP(m)) having generator

$$\mathcal{L}^{BEP(m)} = \sum_{i=1}^{L-1} z_i z_{i+1} (\partial_i - \partial_{i+1})^2 - \frac{m}{2} (z_i - z_{i+1}) (\partial_i - \partial_{i+1}) \quad (16)$$

For the BEP(m) process the total energy $\sum_{i=1}^L Z_i(t)$ is conserved. It can be proved that the stationary reversible measure are product with marginals $Gamma\left(\frac{m}{2}, \theta\right)$, i.e. with probability density

$$p(z) = \prod_{i=1}^L \frac{z_i^{\frac{m}{2}-1} e^{-\frac{z_i}{\theta}}}{\Gamma\left(\frac{m}{2}\right) \theta^{\frac{m}{2}}} \quad (17)$$

Exploiting a change of representation of the SU(1,1) algebra from

$$K_i^+ = z_i \quad K_i^- = z_i \partial_i^2 + \frac{m}{2} \partial_i \quad K_i^0 = z_i \partial_i + \frac{m}{4} \quad (18)$$

to

$$K_i^+ |\eta_i\rangle = \left(\eta_i + \frac{m}{2}\right) |\eta_i + 1\rangle \quad K_i^- |\eta_i\rangle = \eta_i |\eta_i - 1\rangle \quad K_i^0 |\eta_i\rangle = \left(\eta_i + \frac{m}{4}\right) |\eta_i\rangle \quad (19)$$

one deduces that the BEP(m) process admits a dual given by the SIP(m) process with generator

$$\begin{aligned} \mathcal{L}^{SIP(m)} f(\eta) &= \sum_i \eta_i \left(\frac{m}{2} + \eta_{i+1}\right) [f(\eta^{i,i+1}) - f(\eta)] \\ &\quad + \eta_{i+1} \left(\frac{m}{2} + \eta_i\right) [f(\eta^{i+1,i}) - f(\eta)] \end{aligned} \quad (20)$$

The following result is a generalization of the duality relation between BMP and SIP models.

Theorem 7.1 (Duality BEP(m)/SIP(m)) *The process with generator $\mathcal{L}^{BEP(m)}$ in (16) and the process with generator $\mathcal{L}^{SIP(m)}$ in (20) are dual with duality function*

$$D(z, \eta) = \prod_{i=1}^L z_i^{\eta_i} \frac{\Gamma\left(\frac{m}{2}\right)}{2^{\eta_i} \Gamma\left(\frac{m}{2} + \eta_i\right)}.$$

7.2 Instantaneous thermalization limit

The idea behind instantaneous thermalization limit is to imagine that the process evolves through jumps and at each jump it immediately thermalize with respect to the invariant measure. For the BEP(m) this leads to the following definition for the generator of the instantaneous thermalization limit on the edge $(i, i+1)$

$$\begin{aligned} \mathcal{L}_{i,i+1}^{IT,BEP(m)} f(z_i, z_{i+1}) &:= \lim_{t \rightarrow \infty} \left(\left(e^{tL^{BEP(m)}} - 1 \right) f \right) (z_i, z_{i+1}) = \\ &\int \rho^m(z'_i, z'_{i+1} | z'_i + z'_{i+1} = z_i + z_{i+1}) \left[f(z'_i, z'_{i+1}) - f(z_i, z_{i+1}) \right] dz'_i dz'_{i+1} \end{aligned}$$

where $\rho^{(m)}$ denotes the probability density of the stationary state of the BEP(m) process on two sites. Since we know that for two independent random variables (Z_1, Z_2) with distribution $Gamma\left(\frac{m}{2}, \theta\right)$ the ratio $P = Z_1/(Z_1 + Z_2)$ is distributed like $Beta\left(\frac{m}{2}, \frac{m}{2}\right)$ then we can also write

$$\begin{aligned} \mathcal{L}_{i,i+1}^{IT,BEP(m)} f(z_i, z_{i+1}) &= \\ &\int_0^1 dp \nu^{(m)}(p) \left[f(p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1})) - f(z_i, z_{i+1}) \right] \end{aligned}$$

where $\nu^{(m)}$ denotes the probability density of $Beta\left(\frac{m}{2}, \frac{m}{2}\right)$ distribution. In particular, since $Beta(1, 1)$ coincides with the uniform distribution, then for $m = 2$ the generator of the KMP process [16] is recovered, i.e.

$$\mathcal{L}_{i,i+1}^{IT,BEP(2)} f(z_i, z_{i+1}) = \mathcal{L}_{i,i+1}^{KMP} f(z_i, z_{i+1}) \quad (21)$$

where

$$\mathcal{L}_{i,i+1}^{KMP} f(z_i, z_{i+1}) = \int_0^1 dp \left[f(p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1})) - f(z_i + z_{i+1}) \right]$$

8 Duality and multiple conservation laws

In this last chapter we present an example of a diffusion process that conserves (in the bulk) its total energy and momentum (see also [2, 3]). We investigate its duality relations and, last, we infer a general theorem about duality and change of coordinates. The results discussed in this section are taken from [7].

8.1 A diffusion process with conservation of energy and momentum

The basic process we examine is a Markov process defined by its generator as follows.

Definition 8.1 (of the process) Consider a diffusion process $(\mathbf{X}_t)_{t \geq 0} = (X_t, Y_t, Z_t)_{t \geq 0}$ taking values in \mathbb{R}^3 . The vector (x, y, z) represents the momentum associated with three unit mass particles $\{1, 2, 3\}$ freely moving in a physical volume V . Thus, up to irrelevant constant, the total momentum of the system is $P = x + y + z$ and the total (kinetic) energy is $E = x^2 + y^2 + z^2$. The generator of the process is

$$\mathcal{L} = [(x\partial_y - y\partial_x) + (y\partial_z - z\partial_y) + (z\partial_x - x\partial_z)]^2 \quad (22)$$

where we shorthand $\partial_i = \frac{\partial}{\partial i}$ with $i \in \{x, y, z\}$. \mathcal{L} acts on twice differentiable functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$.

A distinguishing property of this process regards the conservation of both total energy E and total momentum P . This property can be easily proved by letting act the generator on those functions. It is easy to see that they are both zero. The same result can also be achieved via the stochastic differential equations (in Itô sense) of \mathcal{L} associated to generator (22), which are

$$\begin{cases} dx_t = (-2x_t + y_t + z_t)dt + (y_t - z_t)dB_t \\ dy_t = (x_t - 2y_t + z_t)dt + (z_t - x_t)dB_t \\ dz_t = (x_t + y_t - 2z_t)dt + (x_t - y_t)dB_t. \end{cases} \quad (23)$$

By inspection it is possible to find that the total momentum P is conserved

$$\begin{aligned} d(x_t + y_t + z_t) &= (-2x_t + y_t + z_t)dt + (y_t - z_t)dB_t + (x_t - 2y_t + z_t)dt + \\ &\quad (z_t - x_t)dB_t + (x_t + y_t - 2z_t)dt + (x_t - y_t)dB_t = 0 \end{aligned}$$

and, by making use of the Itô's formula, the total energy of the process E satisfies

$$\begin{aligned} d(x_t^2 + y_t^2 + z_t^2) &= [2x_t(-2x_t + y_t + z_t) + 2y_t(x_t - 2y_t + z_t) + 2z_t(x_t + y_t - 2z_t) \\ &\quad + 2(y_t - z_t)^2 + 2(z_t - x_t)^2 + 2(x_t - y_t)^2] dt \\ &\quad + 2[x_t y_t - x_t z_t + y_t z_t - y_t x_t + z_t x_t - z_t y_t] dB_t \\ &= 0. \end{aligned}$$

Before going any further, we want to highlight the geometric aspects of our problem. Generator (22) can be viewed as the result of three different rotations. To be more specific, $(x\partial_y - y\partial_x)$ generates the rotation across the z -axis, $(y\partial_z - z\partial_y)$ generates the rotation across the x -axis and $(z\partial_x - x\partial_z)$ generates the rotation across the y -axis, so it turns out that generator (22) represents the rotation around the $(1, 1, 1)$ -axis, which is orthogonal to the plane of equation $x + y + z = \text{constant}$.

As a consequence of the conservation of both total energy and total momentum, the motion takes place in the 1-dimensional manifold (i.e. a circle) given by the intersection of the sphere $x^2 + y^2 + z^2 = E$ and the plane $x + y + z = P$ orthogonal to the rotation axis just mentioned.

Remark 8.2 (Extended system) *It is easy to define a one-dimensional system with L sites coupled with two external reservoirs at different temperatures T_l and T_r that conserve in the bulk both energy and momentum. The generator of this process is given by the sum of each generator of the type of (22) for the bulk part, where the contribution of the two reservoirs has been added:*

$$\mathcal{L} = \mathcal{L}_l + \sum_{i=1}^{L-2} \mathcal{L}_{i,i+1,i+2} + \mathcal{L}_r. \quad (24)$$

8.2 Duality results

One might wonder about the existence of a dual process in the setting of multiple conservations laws. In order to find a duality relation we make a change of coordinates that simplifies the expression of the generator (22). This is achieved by a rotation that maps the $(1, 1, 1)$ axis to the z' axis of the new coordinates system. In other words, we need to find matrix R such that

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Matrix R describes the rotation that moves a frame $Ox'y'z'$, initially aligned with $Oxyz$, into a new orientation in which the Oz' axis is brought into the $(1, 1, 1)$ axis. This matrix is easily found through Euler angles

$$R = \begin{pmatrix} -\frac{\sqrt{2}}{2}\cos\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\sin\varphi & \frac{\sqrt{2}}{2}\sin\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\cos\varphi & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{2}\cos\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\sin\varphi & -\frac{\sqrt{2}}{2}\sin\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\cos\varphi & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}}\sin\varphi & \frac{\sqrt{2}}{\sqrt{3}}\cos\varphi & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (25)$$

This change of coordinates let us find the generator of the process that conserves total momentum and energy (and we call it \mathcal{L}'_3 to highlight the fact that 3 sites are involved) as function of x' , y' and z'

$$\mathcal{L}'_3 = [(x\partial_y - y\partial_x) + (y\partial_z - z\partial_y) + (z\partial_x - x\partial_z)]^2 = 3 [x'\partial_{y'} - y'\partial_{x'}]^2. \quad (26)$$

To make it clearer we call the latter equation above \mathcal{L}'_3 , i.e. the generator of the Brownian Momentum process with 2 sites $(x\partial_y - y\partial_x)^2$ for which duality is verified, as shown in [11].

Thereby the duality function is obtained by the knowledge of

$$D'(x', y'; n_1, n_2) = d(x'; n_1)d(y'; n_2) = \frac{x'^{2n_1}y'^{2n_2}}{(2n_1 - 1)!!(2n_2 - 1)!!}$$

which is the duality function of the process of generator \mathcal{L}'_3 .

Thanks to matrix R in (25), it is possible to write x' and y' as functions of x , y , z and consequently one

finds the duality function for our process:

$$\begin{aligned}
D(x, y, z; n_1, n_2) = & \tag{27} \\
& \frac{[(-\frac{\sqrt{2}}{2}\cos\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\sin\varphi)x + (\frac{\sqrt{2}}{2}\cos\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\sin\varphi)y + \frac{\sqrt{2}}{\sqrt{3}}\sin\varphi z]^{2n_1}}{(2n_1 - 1)!!} \\
& \cdot \frac{[(\frac{\sqrt{2}}{2}\sin\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\cos\varphi)x + (-\frac{\sqrt{2}}{2}\sin\varphi - \frac{\sqrt{2}}{2\sqrt{3}}\cos\varphi)y + \frac{\sqrt{2}}{\sqrt{3}}\cos\varphi z]^{2n_2}}{(2n_2 - 1)!!}.
\end{aligned}$$

Remark 8.3 *Due to the arbitrary of φ , this example shows that there are infinitely many duality functions not trivially related by a multiplicative factor.*

The non-trivial duality relation discussed in the previous section is formalized in the following theorem.

Theorem 8.4 *The process $\mathbf{X}(t) = (x(t), y(t), z(t))$ with generator \mathcal{L}_3 in equation (26) is dual to the process $\mathbf{N}(t) = (n_1(t), n_2(t))$, with generator*

$$\begin{aligned}
L_3 f(n_1, n_2) = & n_1 \left(n_2 + \frac{1}{2} \right) [f(n_1 - 1, n_2 + 1) - f(n_1, n_2)] \\
& + n_2 \left(n_1 + \frac{1}{2} \right) [f(n_1 + 1, n_2 - 1) - f(n_1, n_2)]
\end{aligned}$$

on duality function $D = D(x, y, z; n_1, n_2)$ in (27).

Proof. The result is proved by an explicit computation that shows that the definition of duality in (11) is satisfied. □

8.3 Duality theory under change of coordinates

The purpose of this last section is to extend for a generic situation the duality result found for the process that leaves energy and momentum invariant. Starting from a known duality and after a change of coordinates, we ask about the duality properties in the new system of coordinates. The main idea is explained in Figure 2, where full arrows allude to a duality relation: D' represents the known duality function, while D is the new one. The relation given by dashed arrow R_φ , as formalized in theorem 8.5, is between the same function spaces, for which we use two different coordinate systems.

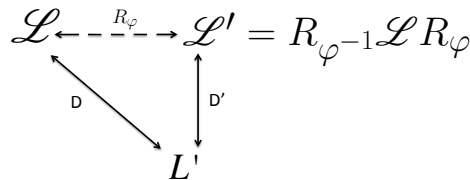


Figure 2: A schematic representation of theorem 8.5, full arrows indicate a duality relation, while dashed map R_φ is a relation between two function spaces.

Theorem 8.5 (Duality and change of coordinates) Consider the function spaces $\mathcal{B}(\Omega) = \{f : \Omega \rightarrow \mathbb{R}\}$ and $\mathcal{B}(\Omega') = \{f : \Omega' \rightarrow \mathbb{R}\}$, where Ω and Ω' are two generic spaces. Let

$$R_\varphi : \mathcal{B}(\Omega') \rightarrow \mathcal{B}(\Omega)$$

$$f \rightarrow (R_\varphi f)(x) = (f \circ \varphi)(x)$$

the composition with function

$$\varphi : \Omega \rightarrow \Omega'$$

$$x \rightarrow \varphi(x) = x',$$

where we assume that φ is an invertible function.

Consider also the two operators $\mathcal{L}' : \mathcal{B}(\Omega') \rightarrow \mathcal{B}(\Omega')$ and $L' : \mathcal{B}(\Omega^{dual}) \rightarrow \mathcal{B}(\Omega^{dual})$ dual to each other with duality function $D'(x', n') : \Omega' \times \Omega^{dual} \rightarrow \mathbb{R}$, i.e.,

$$[\mathcal{L}' D'(\cdot, n')](x') = [L' D'(x', \cdot)](n'). \quad (28)$$

Let $\mathcal{L} : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$ be an operator related to \mathcal{L}' as follows

$$\mathcal{L}' = R_{\varphi^{-1}} \mathcal{L} R_\varphi. \quad (29)$$

Then, operator \mathcal{L} is dual to L' through $D : \Omega \times \Omega^{dual} \rightarrow \mathbb{R}$, i. e.

$$[\mathcal{L} D(\cdot, n')](x) = [L' D(x, \cdot)](n') \quad (30)$$

where the duality function D is

$$D(x, n') := (R_\varphi D'(\cdot, n'))(x) = D'(\varphi(x), n'). \quad (31)$$

Proof. Consider the following chain of equalities given by (28), (29) and (31)

$$L' D(x, \cdot) = \mathcal{L}' D'(\cdot, n') = R_{\varphi^{-1}} \mathcal{L} R_\varphi D'(\cdot, n') = R_{\varphi^{-1}} \mathcal{L} D(x, \cdot)$$

Multiplying each side for R_φ we have

$$\mathcal{L} D(x, \cdot) = R_\varphi L' D'(\cdot, n') = L' R_\varphi D'(\cdot, n') = L' D(x, \cdot)$$

□

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