Global solutions of a free boundary problem via mass transport inequalities

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Abstract

We study a free boundary problem which arises as the continuum version of a stochastic particles system in the context of Fourier law. Local existence and uniqueness of the classical solution are well known in the literature of free boundary problems. We introduce the notion of generalized solutions (which extends that of classical solutions when the latter exist) and prove global existence and uniqueness of generalized solutions for a large class of initial data. The proof is obtained by characterizing a generalized solution as the unique element which separates suitably defined lower and upper barriers in the sense of mass transport inequalities.

1 Introduction and results

Free boundary problems (FBP) for the linear heat equation have been investigated since the discovery of the law of heat conduction by J.B.J. Fourier [23]. A celebrated example is the Stefan problem, i.e. the description of the interface between ice and water in the presence of a first-order phase transition [31].

We study a variant of the Fourier law by considering a one dimensional system where total mass is conserved and a same density gradient is prescribed at the boundaries (which physically means that we are injecting and removing mass at same rate). The right boundary is let free to move and this gives rise to a FBP whose analysis is the content of this work. As we shall see the space derivative of the solution is related to the classical Stefan problem. In one dimension the theory of Stefan FBP is very rich and detailed results are available [13, 14, 21, 27, 32].

In particular, under appropriate assumptions on the initial datum, global (in time) existence and uniqueness theorems are known. However in the general case only existence results local in time are available due to the appearance of singularities for the classical solutions.

The aim of this paper is to study the extension of the FBP solution past the singularities. By a combination of probabilistic methods and results in the context of mass transport theory we provide *global* existence and uniqueness. The result for all time is achieved by introducing a new notion of *generalized* solution obtained as the weak limit of the classical solutions of an auxiliary FBP problem with a relaxed mass constraint. The scheme is applicable to a large set of initial conditions, including bounded and integrable initial data.

1.1 The free boundary problem.

Motivated by recent papers (see [2, 3, 9, 10, 11, 12]) on particles systems which approximate free boundary problems we study here the macroscopic version of the model introduced in [2]. In Definition 1.1 below we give its classical formulation.

Definition 1.1 (The FBP 1.1)**.** The pair $(X_t, \rho(\cdot,t))$ is a classical solution of the FBP 1.1 in the time interval $[0,T)$ with initial datum $(X_0, \rho_0(\cdot))$ if it satisfies: i) $X_t \in C^1([0,T),\mathbb{R}_+)$ is strictly positive and $X_{t=0} = X_0$; ii) for each $t \in [0,T)$, $\rho(\cdot,t) \in C^2((0,X_t),\mathbb{R}_+)$ and it has limits with its derivatives at 0 and X_t; moreover for each $r \in [0,X_t]$, $\rho(r,t)$ is differentiable in t; iii) the following equations (with j a positive parameter) are pointwise satified

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}(r,t) &= \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}(r,t) && t \in [0,T), r \in (0,X_t) && (1.1)\\
\rho(r,0) &= \rho_0(r), && r \in (0,X_0) && (1.2)\\
\frac{\partial \rho}{\partial r}(0,t) &= -2j && t \in [0,T) && (1.3)\\
\rho(X_t,t) &= 0 && t \in [0,T) && (1.4)\\
\frac{\partial \rho}{\partial r}(X_t,t) &= -2j && t \in [0,T) && (1.5)
\end{aligned}
$$

By (1.2) the initial data $(X_0, \rho_0(\cdot))$ of classical solutions must be as regular as above and satisfy (1.3)–(1.5) with $t = 0$.

Remark 1.1. $\rho(\cdot,t)$ is a non-negative function that we interpret as a mass density which is concentrated in the time-varying interval $[0,X_t]$. X_t is then the right edge of the mass distribution and it is called free boundary because it is itself part of the problem.

The condition (1.3) describes a constant incoming mass flow (represented by $-\frac{1}{2}\frac{\partial\rho}{\partial r}(0,t)$) through the origin, with mass injected at rate $j > 0$, and the condition (1.5) states that there is an outgoing mass flow, $-\frac{1}{2}\frac{\partial\rho}{\partial r}(X_t,t)$, through X_t which is also equal to j. Inside $(0,X_t)$ the mass diffuses freely according to the linear heat equation (1.1), thus in a classical solution the total mass is conserved

$$\int_0^{X_t} \rho(r,t)\,dr = \int_0^{X_0} \rho(r,0)\,dr \qquad t \in (0,T)\ . \tag{1.6}$$

This can be formally seen by differentiating the left hand side and using (1.1)—(1.5); moreover one can check that classical solutions of FBP 1.1 coincide with classical solutions of the FBP problem that is obtained replacing condition (1.5) by (1.6).

Remark 1.2. The physics behind the model recalls the Fourier's law where heat is injected from one side and removed from the other. This suggests that stationary solutions are the functions

$$(X_t, \rho(r,t)) := \left(\frac{a}{2j}, \rho_a(r)\right)$$

where $a > 0$ and

$$\rho_a(r) = a - 2jr, \qquad 0 \le r \le \frac{a}{2j}\ . \tag{1.7}$$

which are indeed classical stationary solutions of (1.1)—(1.5). Unlike in the Fourier's law here we have a whole family of stationary solutions, this is because we do not fix the density at the endpoints but only the current: physically this means that we deal with a "current reservoir" while in the Fourier law we have thermal reservoirs (which fix the temperatures at the endpoints).

We shall later use the functions $\rho_a(r)$ to bound the solutions of the problem (1.1)—(1.5) via mass transport inequalities.

Local existence of classical solutions of FBP 1.1 follows from the literature of Stefan problems:

Theorem 1.1 (Local existence)**.** *Suppose the initial datum (X_0, ρ_0) satisfies: $X_0 > 0$, $\rho_0 \in C^3([0, X_0), \mathbb{R}_+)$, ρ_0 has limit with its derivatives at X_0 and $\rho_0(X_0) = 0$, $\frac{d\rho_0}{dr}(X_0) = -2j$, $\frac{d\rho_0}{dr}(0) = -2j$. Then there exists $T > 0$ and a classical solution $(X_t, \rho(\cdot, t))$, $t \in [0, T)$, with initial datum (X_0, ρ_0).*

Proof. Consider the classical Stefan problem in $t \in [0, T), r \in (0, X_t)$:

$$\begin{aligned}
&\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial r^2} \qquad v(r,t)\Big|_{r=0,X_t} = 0, \quad v(r,0) = -\frac{1}{2}\frac{\partial \rho_0(r)}{\partial r} - j \\
&\frac{dX_t}{dt} = -(2j)^{-1}\frac{\partial v(r,t)}{\partial r}\Big|_{r=X_t}
\end{aligned} \tag{1.8}$$

which is formally obtained from (1.1)—(1.5) by setting $v(r,t) := -\dfrac{1}{2}\dfrac{\partial \rho}{\partial r}(r,t) - j$; the equation for X_t being obtained by differentiating the identity $\rho(X_t, t) = 0$. Local existence for (1.8) is proved in [14]–[18].

Given X_t and $v(r,t)$ satisfying (1.8) we set

$$\rho(r,t) = 2\int_r^{X_t} \Big(v(r',t) + j\Big) dr' \tag{1.9}$$

and check that (1.1)—(1.5) are satisfied by $(X_t, \rho(\cdot, t))$. Non negativity of $\rho(\cdot, t)$ follows from the maximum principle, see also Section 4 where we express $\rho(\cdot, t)$ in terms of Green functions. □

Uniqueness of the local classical solution for the Stefan problem (1.8) is also known in the literature, however this does not immediately imply uniqueness for FBP 1.1 which will be proved in Theorem 1.2.

Following Fasano and Primicerio (see e.g. [14]) we say that if $v(r,0) \geq 0$ then (1.8) has a "sign specification". With a sign specification the solution is global while if there is no sign specification in general we only have local existence with examples where singularities do appear. The analysis of their structure is a very interesting and much studied problem, see for instance [7], [19], [20], [28].

Our main goal in this paper is the extension of the solution past the singularities. We thus want to introduce a notion of "generalized solutions" which reduces to classical solutions in the smooth case and discuss global existence and uniqueness of such generalized solutions.

1.2 Quasi-solutions and generalized solutions.

We define a generalized solution of the FBP 1.1 as the weak limit of solutions of approximate problems calling the latter "quasi-solutions".

Definition 1.2 (Quasi-solutions)**.** Let $\rho_0 \in L^\infty(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$ be such that:

$$0 < R(\rho_0) := \inf \Big\{ r : \int_r^\infty \rho_0(r')\, dr' = 0 \Big\} < \infty \tag{1.10}$$

Then $(X_t, u(\cdot,t), \epsilon)$, $t \in [0,T]$, $T > 0$, $\epsilon > 0$, is a quasi-solution of the FBP 1.1 in the time interval $[0,T)$ with initial datum ρ_0 and accuracy parameter ϵ if the following conditions are satisfied:

- X_t is strictly positive, Lipschitz continuous and piecewise C^1 (with finitely many discontinuities of the derivative)
- $u(r,t)$, $t \in [0,T)$, $r \in [0, X_t]$, is smooth (in the sense of (ii) of Definition 1.1) and it solves (1.1), (1.3), (1.4) for all $t \in (0,T)$.
- The condition (1.2) is replaced by $\displaystyle\int |u(r,0) - \rho_0(r)| dr \leq \epsilon$.
- The condition (1.6) is replaced by

$$\sup_{t \leq T} \Big| \int_0^{X_t} u(r,t)\, dr - \int_0^{X_0} u(r,0)\, dr \Big| \leq \epsilon\ . \tag{1.11}$$

A sequence $\{(X_t^{(n)}, \rho^{(n)}(\cdot,t), \epsilon_n),\ t \in [0,T]\}$ of quasi-solutions in $[0,T)$ with initial datum ρ_0 is called "optimal" if $\lim_{n\to\infty} \epsilon_n = 0$.

Definition 1.3 (Generalized solutions)**.** Let ρ_0 be as in Definition 1.2 and $T > 0$. Then $\rho(r,t)$, $r \in \mathbb{R}_+$, $t \in [0,T)$, is a generalized solution in $[0,T)$ with initial datum ρ_0 of the FBP 1.1 if there exists an optimal sequence $(X_t^{(n)}, \rho^{(n)}(\cdot,t), \epsilon_n)$, $t \in [0,T]$, of quasi-solutions in the time interval $[0,T)$ with initial datum ρ_0 such that

$$\lim_{n\to\infty} \rho^{(n)} = \rho \quad \text{weakly}$$

The main result in this paper is a proof of *global existence and uniqueness of generalized solutions with a variational representation of the generalized solution as the unique separating element of lower and upper barriers.* This will be explained in Section 1.3 below.

Theorem 1.2 (Existence and uniqueness). *For any ρ_0 as in Definition 1.2 and any $T > 0$ the following holds.*

(a) There exists an optimal sequence of quasi-solutions in $[0, T)$ with initial datum ρ_0.

(b) Any optimal sequence of quasi-solutions in $[0, T)$ with initial datum ρ_0 converges weakly to a limit which is (by definition) a generalized solution of the FBP 1.1.

(c) There exists a function $\bar{\rho} = \{\bar{\rho}(r, t),\ r \in \mathbb{R}_+,\ t \in \mathbb{R}_+\}$, continuous in (r, t), such that if $u(\cdot, t)$ is a generalized solution of the FBP 1.1 in $[0, T)$ with initial datum ρ_0 then $u(\cdot, t) = \bar{\rho}(\cdot, t)$ for all $t \in [0, T)$.

Remark 1.3. From Theorem 1.2 we thus conclude that all optimal sequences of quasi-solutions (in any interval $[0, T)$ and with initial datum ρ_0) converge to the function $\bar{\rho}$, which is then the unique generalized solution. Moreover since any classical solution of the FBP 1.1 is also an optimal sequence of quasi-solutions with $\epsilon_n \equiv 0$ in the time interval where it exists then it is also a generalized solution. Thus when the unique generalized solution is smooth (in the sense of (ii) of Definition 1.1) it is a classical solution.

Finally observe that the FBP 1.1 can be simulated by the stochastic particle evolution studied in [2] which provides a discrete approximation convergent in the macroscopic hydrodynamic limit.

In the next section we give a characterization of the function $\bar{\rho}$ as the unique separating element between barriers.

1.3 A variational representation of the evolution.

The key to the proof of Theorem 1.2 is variational as it exploits the monotonicity properties (in the sense of mass transport) of the FBP 1.1 by introducing lower and upper barriers.

The notion of barriers for the construction of solutions of partial differential equations is well known [26, 22] (see also [8] in the context of motion by mean curvature). In our case upper and lower barriers will provide upper and lower bounds with respect to an order based on mass transport.

We construct the barriers by quantizing the way to add and remove mass to the system: we add quanta of mass ($= j\delta$) at the origin at discrete times, say $k\delta$, $\delta > 0$, $k \in \mathbb{N}$, and simultaneously we remove the same amount of mass at a left neighbor of the edge. Since at each step a finite mass is added all at the origin we shall need to enlarge the functional space we work with to include the Dirac delta (cfr. Definition 1.4). In the time intervals $(k\delta, (k+1)\delta)$ we evolve using the linear heat equation on $\mathbb{R}_+$ with Neumann condition at 0. We will obtain the upper barrier if the addition/removal starts at time 0, and the lower barrier if it starts at time δ. We shall prove, see Theorem 1.3, that there is a unique element which separates the upper and lower barriers constructed in this way. The discrete scheme by which the barriers are defined was previously used in [2] and introduced in [9] to study a similar particle model.

We define the barriers in Definition 1.7 below after some preliminaries.

Definition 1.4. (The space $\mathcal{U}$). $\mathcal{U}$ is the space of positive Borel measures on $\mathbb{R}_+$ which are sum of cD_0, $c \geq 0$, D_0 the Dirac delta at 0, and $\rho\, dr$, with $\rho \in L^\infty(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$. We shall denote its elements by u and by an abuse of notation, write $u = c_u D_0 + \rho_u$. For u and v in $\mathcal{U}$ we call

$$|u - v| = |c_u - c_v| D_0 + |\rho_u - \rho_v|, \quad |u - v|_1 = |c_u - c_v| + \int |\rho_u - \rho_v|\, dr \tag{1.12}$$

We further define

$$F(r; u) = c_u \mathbf{1}_{\{0\}}(r) + \int_r^\infty \rho_u(r')\, dr' \tag{1.13}$$

where $\mathbf{1}_{\{0\}}(r) = 0$ unless $r = 0$, in which case it is equal to 1. Note that $F(0; u)$ is the total mass of the measure u and $F(0; |u - v|) = |u - v|_1$ is the total variation norm of $u - v$. We call $\mathcal{U}_\delta$, $\delta \geq 0$, the following subset of $\mathcal{U}$:

$$\mathcal{U}_\delta := \Big\{ u \in \mathcal{U} : \ F(0; \rho_u) > j\delta \Big\} \tag{1.14}$$

Remark 1.4. Observe that $F(r; u)$ is a non increasing function of r which starts at 0 from $F(0; u)$ which is the total mass of u. If u has compact support $F(r; u) = 0$ definitively and, in agreement with (1.10), we can define the "edge" $R(u)$ as

$$R(u) := \inf\{r : F(r; u) = 0\} \tag{1.15}$$

Definition 1.5. (The cut and paste operator). The *cut-and-paste* operator $K^{(\delta)} : \mathcal{U}_\delta \to \mathcal{U}$ is defined as

$$K^{(\delta)} u = j\delta D_0 + \mathbf{1}_{[0, R_\delta(u)]} u, \qquad R_\delta(u) := \inf\{r : F(r; u) = j\delta\} \tag{1.16}$$

Definition 1.6. (The free evolution). Call $G_t^{\text{neum}}(r, r')$, $r, r' \in \mathbb{R}_+$ the Green function of the heat equation in $\mathbb{R}_+$ with Neumann boundary conditions at 0, namely

$$G_t^{\text{neum}}(r, r') = G_t(r, r') + G_t(r, -r'), \quad G_t(r, r') = \frac{e^{-\frac{(r-r')^2}{2t}}}{\sqrt{2\pi t}} \tag{1.17}$$

Observing that $G_t^{\text{neum}}(r, r') = G_t^{\text{neum}}(r', r)$ we write for $u \in \mathcal{U}$:

$$G_t^{\text{neum}} * u(r) = \int_{\mathbb{R}_+} G_t^{\text{neum}}(r, r') u(r')\, dr' = \int_{\mathbb{R}_+} G_t^{\text{neum}}(r', r) u(r')\, dr'$$

Definition 1.7. (Barriers). Let $u \in L^\infty(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$ and such that $F(0; u) > 0$. Then for all δ small enough $u \in \mathcal{U}_\delta$ and for such δ we define the "barriers" $S_{k\delta}^{(\delta,\pm)}(u)$, $k \in \mathbb{N}$, as follows: we set $S_0^{(\delta,\pm)}(u) = u$, and, for $k \geq 1$,

$$S_{k\delta}^{(\delta,-)}(u) = K^{(\delta)} G_\delta^{\text{neum}} * S_{(k-1)\delta}^{(\delta,-)}(u) \qquad (1.18)$$
$$S_{k\delta}^{(\delta,+)}(u) = G_\delta^{\text{neum}} * K^{(\delta)} S_{(k-1)\delta}^{(\delta,+)}(u)$$

The functions $S_{k\delta}^{(\delta,\pm)}(u)$ deserve the name of "lower and upper barriers" because we shall see that they form separated classes with respect to the following notion of partial order:

Definition 1.8. (Partial order). For any $u, v \in \mathcal{U}$ we set

$$u \leq v \quad \text{iff} \quad F(r; u) \leq F(r; v) \quad \text{for all } r \geq 0 \qquad (1.19)$$

with $F(r; u)$ as in (1.13).

When u and v have the same total mass then $u \leq v$ if and only if v can be obtained from u by moving mass to the right, this statement will be made precise in Proposition 2.2.

In Proposition 3.1 we will prove many properties of the barriers and in particular that they are monotone (namely the lower barrier is non increasing and the upper one is non decreasing) and that for all $\delta > 0$

$$S_{k\delta}^{(\delta,-)}(u) \leq S_{k\delta}^{(\delta,+)}(u)$$

Thus upper and lower barriers form separated classes and in the following Theorem we prove that there is a unique separating element.

Theorem 1.3 (Barriers and separating elements). *Let $u \in L^\infty(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$ with $R(u) < \infty$, then there exists a unique function $S_t(u)(r)$ continuous in (r, t) for $t > 0$ such that for all $t > 0$ and $r \in \mathbb{R}_+$:*

$$F(r; S_t(u)) = \lim_{\ell \to \infty} F(r; S_t^{(2^{-\ell}t,\pm)}(u)) \quad \text{monotonically} \qquad (1.20)$$

$$F(r; S_t(u)) = \inf_{\delta: t = k\delta, k \in \mathbb{N}} F(r; S_t^{(\delta,+)}(u)) = \sup_{\delta: t = k\delta, k \in \mathbb{N}} F(r; S_t^{(\delta,-)}(u)) \qquad (1.21)$$

Moreover $S_t(u) \to u$ weakly as $t \to 0$ and if $u \leq v$ then $S_t(u) \leq S_t(v)$.

We have more detailed properties on the structure of the barriers which are stated in Section 3.

A key result of this paper is the identification of the separating element between barriers with the generalized solution of the FBP 1.1, namely with the function $\bar{\rho}$ of Theorem 1.2.

Theorem 1.4 (Characterization of $\bar{\rho}$). *The generalized solution $\bar{\rho}(\cdot, t)$ of the FBP 1.1 with initial datum ρ_0 as in Definition 1.2 is equal to the separating element $S_t(\rho_0)$ of Theorem 1.3. Moreover there is $a_2 < \infty$ (which depends on ρ_0) so that*

$$R(\bar{\rho}(\cdot, t)) < a_2 \qquad \text{for all} \quad t \geq 0,$$

and if ρ_0 is strictly positive in a neighborhood of the origin then there exists $a_1 > 0$ so that

$$R(\bar{\rho}(\cdot, t)) > a_1 \qquad \text{for all} \quad t \geq 0.$$

1.4 Scheme of the proofs

The proofs of the results exploit analytical and probabilistic arguments and are organized in the following way.

In Section 2 we prove mass transport inequalities (some well known in the literature) which are crucial in the subsequent steps. These inequalities hold true with respect to the notion of partial order in Definition 1.8 and with their help we deduce in Section 3 properties of the barriers which lead to the proof of Theorem 1.3. In particular we show that the sequence of upper and lower barriers admits a unique limit that separates them.

In Section 4 we recall the representation of the solution of the heat equation in terms of Brownian motions and use it to prove part (a) of Theorem 1.2, namely the existence of an optimal sequence of quasi-solutions.

In Section 5 we use the uniqueness of the separating element between barriers to reduce the proofs of part (b), (c) of Theorem 1.2 and Theorem 1.4 to the variational problem of showing that quasi-solutions are in between lower and upper barriers (modulo a "small error"). The crucial step is Proposition 5.1 in Section 5.1 where we prove that any quasi-solution $(X_t, \rho(\cdot,t), \epsilon)$, $t \in [0,T]$ is (up to a "small error") in between the barriers that start at $t = 0$ from $\rho(\cdot, 0)$. As a consequence of this result we prove in Section 5.4 that:

- any optimal sequence $(X_t^{(n)}, \rho^{(n)}(\cdot,t), \epsilon_n)$, $t \in (0,T)$ of quasi-solutions with initial datum ρ_0 is such that the sequence of measures $\rho^{(n)}(r,t)dr$ on $\mathbb{R}_+$ is tight;
- any weak limit of $\rho^{(n)}(\cdot,t)$ is in between the barriers $S_t^{(2^{-\ell}t,\pm)}(\rho_0)$;
- by letting $\ell \to \infty$, and using Theorem 1.3 we then conclude that any weak limit is equal to $S_t(\rho_0)$, thus getting the statements (b) and (c) of Theorem 1.2;
- the proof of Theorem 1.4 is then a corollary of all this and it is given at the end of Section 5.4.

1.5 Remarks

Generalized solutions. We thus have global existence of generalized solutions, hence a way to continue classical solutions past their singularity times (if they exist). However we can only say that generalized solutions $\rho(r,t)$ are (r,t) continuous and we do not know much about the motion of the edge. In the existence part of the proof of Theorem 1.2 we construct a sequence of quasi-solutions with the edge which moves piecewise linearly (further work would be required to smoothen out the discontinuities of its velocity) but we do not know about the regularity of the motion of the edge in the limit. In Theorem 1.4 we prove that at all times the edge stays strictly positive and it does not drift away to infinity. However, if the edge and $\rho(r,t)$ are "regular" in some time interval then by Theorem 1.4 in that interval they define a classical solution.

FBP with mass conservation. The physics behind our problem is not the same as in the Stefan problem where the existence of a boundary is related to an interface between two phases. Here instead it comes from the requirement of mass balance. A more general formulation of the

FBP with mass conservation would be the following. Find pairs $(X_t, \rho(r,t))$, where $t \geq 0$, $X_t > 0$ and $\rho(r,t)$ is for each $t \geq 0$ a non negative function which solves the problem

$$\begin{aligned} &\frac{\partial \rho}{\partial t}(r,t) = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}(r,t) + f(r), \qquad r \in (0, X_t) \\ &\rho(X_t,t) = 0, \qquad \frac{\partial \rho}{\partial r}(0,t) = 0, \qquad \int_0^{X_t} \rho(r,t)\,dr = \int_0^{X_0} \rho(r,0)\,dr \end{aligned} \tag{1.22}$$

where $f(r)$ is a non negative function with support in $[0, R_0]$, $R_0 < X_0$. The term $f(r)$ describes a mass source of intensity $\int f(r)dr = j$, with j a positive constant, the mass then diffuses according to the linear heat equation being reflected at 0 due to the Neumann condition, so that it can only escape from X_t where we impose Dirichlet condition. The problem is to determine X_t in such a way that the total mass in conserved.

The FBP 1.1 fits within this scheme by choosing $f = jD_0$, where D_0 denotes the Dirac delta at 0. However our analysis does not extend, at least directly, to smooth f, as in the mass transport inequalities we exploit the fact that mass is added at the leftmost point.

Scaling limits. The FBP 1.1 (or similar versions) appeared in the study of the asymptotic behavior, i.e. the scaling limit, of many different models in the context of statistical mechanics. A non-exhaustive list of examples includes:

Particle systems. The FBP 1.1 has been studied in connection with the Fourier law in a moving domain where ρ is interpreted as an energy density, the sources are reservoirs which add and subtract the same amount of energy at the boundaries while in the bulk energy diffuses according to the linear heat equation. This is discussed in [2] where the above FBP appears as a heuristic guess for the continuum limit of a system of particles. The particles move as independent, symmetric random walks on $\mathbb{N}$ with reflections at 0, new particles are created at rate j at the origin while the rightmost particle is killed also at rate j. To see the reason why the Dirichlet condition appearing in (1.4) corresponds to the killing of the rightmost particle we define $u(r,t) := \rho(r,t)$ for $r \in [0, X_t]$ and $u(r,t) = 0$ elsewhere. Then, in the weak formulation,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} + jD_0 - jD_{X_t}, \qquad r \in \mathbb{R}_+ \tag{1.23}$$

where besides the source at the origin there is also a negative source at X_t (the macroscopic counterpart of the microscopic killing at rate j).

Other particles systems whose scaling limit is given by a FBP of the Stefan type have been studied in [5] and in [9], while – in the reverse direction – reference [25] studied a microscopic particle model for the Stefan freezing/melting problem.

Polymers. In [24] a two-sided version (i.e. with a left and a right free boundaries) of the FBP 1.1 in the interval $[-1, 1]$ without the condition (1.3) (that yields mass conservation) has been studied. The FBP is proved to be the scaling limit (under diffusive scaling) of a random dynamics of polymers with pinning to a substrate. However this problem is substantially different from ours, since mass is not conserved anymore and it is decreasing with time. As a consequence, for Lipschitz initial datum, existence and regularity of the classical solutions could be proved until a maximal time where the two boundaries collide (the proof uses the approach developed in [4]).

Queuing theory. The FBP 1.1 has also natural connections to fluid-limits in queuing theory [30, 1]. It is well known (Burke theorem) that a tandem Jackson network with customers arriving at the first queue as a Poisson process of intensity $\lambda > 0$ gives rise to an outgoing flow of served clients at the last queue that is also Poisson distributed with the same intensity. The hydrodynamic limit of such tandem Jackson network is given by the linear heat equation. The FBP problem studied in this paper is a candidate for the scaling limit of a tandem Jackson network of variable length.

2 Mass transport inequalities

For the reader's convenience we present in this section some well known facts about mass transport and use them to prove some properties which will then be extensively used in the sequel. We start with some elementary properties of the cut-and-paste operator $K^{(\delta)}$ and the diffusion kernel G_δ^{neum}.

Proposition 2.1 (Preliminaries). *Let $\delta > 0$ and let u and v be in $\mathcal{U}_\delta$, then*

$$F(0; u) = F(0; K^{(\delta)}u) = F(0; G_\delta^{\text{neum}} * u) \tag{2.1}$$

$$|K^{(\delta)}u - K^{(\delta)}v|_1 \le |u - v|_1, \quad |G_\delta^{\text{neum}} * u - G_\delta^{\text{neum}} * v|_1 \le |u - v|_1 \tag{2.2}$$

$$|(K^{(\delta)} - 1)u|_1 = 2j\delta \tag{2.3}$$

$$|S_{k\delta}^{(\delta,\pm)}(u) - S_{k\delta}^{(\delta,\pm)}(v)|_1 \le |u - v|_1, \qquad \forall k \in \mathbb{N} \tag{2.4}$$

Proof. (2.1) follows directly from the definition of $K^{(\delta)}$ and G_δ^{neum}. To prove the first inequality in (2.2) we write $u = c_u D_0 + \rho_u$, $v = c_v D_0 + \rho_v$ and assuming that $R_\delta(u) \le R_\delta(v)$,

$$|K^{(\delta)}u - K^{(\delta)}v|_1 = |c_u - c_v| + \int_0^{R_\delta(u)} |\rho_u - \rho_v| + \int_{R_\delta(u)}^{R_\delta(v)} \rho_v$$

We can then add $\int_{R_\delta(v)}^\infty \rho_v$ and subtract $\int_{R_\delta(u)}^\infty \rho_u$ as they are both equal to $j\delta$:

$$|K^{(\delta)}u - K^{(\delta)}v|_1 = |c_u - c_v| + \int_0^{R_\delta(u)} |\rho_u - \rho_v| + \int_{R_\delta(u)}^\infty (\rho_v - \rho_u) \le |u - v|_1$$

Analogously

$$\begin{aligned}|G_\delta^{\text{neum}} * u - G_\delta^{\text{neum}} * v|_1 &\le |c_u - c_v| \int G_\delta^{\text{neum}}(0, r')\, dr' + \int\int |\rho_u(r) - \rho_v(r)| G_\delta^{\text{neum}}(r, r')\, dr\, dr' \\ &= |c_u - c_v| + \int |\rho_u(r) - \rho_v(r)|\, dr\end{aligned}$$

which is equal to $|u-v|_1$. Finally

$$|(K^{(\delta)}-1)u|_1 = j\delta + \int_{R_\delta(u)}^{\infty} \rho_u = 2j\delta$$

while (2.4) is a direct consequence of (2.2). □

Proposition 2.2 (Mass displacement)**.** *Given $u \le v$ in $L^\infty \cap L^1$ with $F(0;u)=F(0;v)$ we define for $r \in \mathbb{R}_+$:*

$$f(r) := \sup \Big\{ r' : \int_0^{r'} v(z)\,dz = \int_0^r u(z)\,dz \Big\}, \tag{2.5}$$

Then

$$f(r) \ge r \tag{2.6}$$

and for any function $\phi \in L^\infty(\mathbb{R}_+,\mathbb{R})$

$$\int_0^\infty v(r)\phi(r)\,dr = \int_0^\infty u(r)\phi(f(r))\,dr \tag{2.7}$$

Proof. Since $F(0;u)=F(0;v)$,

$$\int_0^r u(z)\,dz + F(r;u) = \int_0^r v(z)\,dz + F(r;v)$$

and since $F(r;u) \le F(r;v)$

$$\int_0^r u(z)\,dz \ge \int_0^r v(z)\,dz$$

which yields (2.6). By a density argument (2.7) follows from (2.6).

□

Corollary 2.3 (Moving mass to the right)**.** *Let $u \le v$ in $L^\infty \cap L^1$ and $F(0;u)=F(0;v)$, then for all bounded, non decreasing functions h on $\mathbb{R}_+$:*

$$\int_0^\infty u(r)h(r)\,dr \le \int_0^\infty v(r)h(r)\,dr \tag{2.8}$$

Proof. Observe that (2.8) is verified by definition for all functions h of the form $\mathbf{1}_{[R,\infty)}$, $R \ge 0$. Its validity for functions h as in the text follows from (2.7) because

$$\int_0^\infty v(r)h(r)\,dr = \int_0^\infty u(r)h(f(r))\,dr$$

and $h(f(r)) \ge h(r)$ by (2.6). □

Lemma 2.4 (Left cut)**.** *Let $u \le v$ and assume that $m := F(0;v) - F(0;u) > 0$ Define $\tilde{R}$ so that $\int_0^{\tilde{R}} v = m$, then*

$$u \le v\,\mathbf{1}_{[\tilde{R},+\infty)} =: \tilde{v}, \quad F(0;u) = F(0;\tilde{v}) \tag{2.9}$$

Proof. From $F(0;u) = F(0;v) - m = F(0,\tilde v)$ we get

$$\int_0^r u(z)dz + F(r;u) = \int_0^r v(z)dz + F(r;v) - m$$

Since $F(r;u) \le F(r;v)$, for all $r \ge \tilde R$

$$\int_0^r u(z)dz \ge \int_0^r v(z)dz - m = \int_{\tilde R}^r v(z)dz = \int_0^r \tilde v(z)dz$$

Also for $r < \tilde R$ we have $\int_0^r u \ge 0 = \int_0^r \tilde v$, so that $\int_0^r u \ge \int_0^r \tilde v$ for all r. Since $F(0;u) = F(0;\tilde v)$ the previous inequality implies that $F(r;u) \le F(r;\tilde v)$. $\square$

Definition 2.1 (Partial order modulo m)**.** For any u and v in $\mathcal U$ and $m > 0$, we define

$$u \le v \quad \text{modulo } m \quad \text{iff} \quad \text{for all } r \ge 0\text{:}\ \ F(r;u) \le F(r;v) + m \tag{2.10}$$

Lemma 2.5 (Right cut)**.** *Let $u \le v$ modulo m, then*

$$u^* := u\, \mathbf{1}_{[0,R_m]} \le v \quad \textit{with} \ \ R_m = \inf\{r : F(r;u) = m\} \tag{2.11}$$

Proof. For $r \le R_m$, $F(r;u^*) = F(r;u) - m$ so that $F(r;u^*) \le F(r;v)$. For $r \ge R_m$, $F(r;u^*) = 0 \le F(r;v)$. $\square$

Lemma 2.6 (Partial order under diffusion and cut-and-paste)**.** *Let $u \le v$ in $\mathcal U$, then for any $t > 0$*

$$G_t^{\rm neum} * u \le G_t^{\rm neum} * v \tag{2.12}$$

and if $u \le v$ with u, v in $\mathcal U_\delta$ then

$$K^{(\delta)}u \le u, \quad K^{(\delta)}u \le K^{(\delta)}v, \quad u\, \mathbf{1}_{[0,R_\delta(u)]} \le v\, \mathbf{1}_{[0,R_\delta(v)]} \tag{2.13}$$

Proof. It is clearly enough to prove $G_t^{\rm neum} * u \le G_t^{\rm neum} * \tilde v$ with $\tilde v$ as in (2.9). We have $u \le \tilde v$, $F(0;u) = F(0,\tilde v)$, then, if f is the map defined in Proposition 2.2 relative to u and $\tilde v$, by (2.7) with

$$\phi(r) = \int_R^\infty G_t^{\rm neum}(r,r')\,dr'$$

we get

$$F(R; G_t^{\rm neum} * \tilde v) - F(R; G_t^{\rm neum} * u) = \int u(r)\Big[\phi(f(r)) - \phi(r)\Big]\,dr$$

By an explicit computation: $\frac{d}{dr}\int_R^\infty G_t^{\rm neum}(r,r')\,dr' > 0$, moreover, by Proposition 2.2, $f(r) \ge r$, then (2.12) follows because $\phi(f(r)) \ge \phi(r)$.

The inequality $K^{(\delta)}u \le u$ trivially follows from the definition (1.16). Furthermore we have

$$K^{(\delta)}u - K^{(\delta)}v = (c_u - c_v)D_0 + (\rho_u - \rho_v)\,\mathbf{1}_{[0,R_\delta(u)]} - \rho_v\,\mathbf{1}_{(R_\delta(u),R_\delta(v)]}$$

where $R_\delta(\cdot)$ is defined in (1.16), and $R_\delta(u) \le R_\delta(v)$ because $u \le v$. Hence

$$F(r;K^{(\delta)}u) - F(r;K^{(\delta)}v) = \Big(F(r;u) - F(r;v)\Big)\mathbf{1}_{[0,R_\delta(u)]} - \mathbf{1}_{(R_\delta(u),R_\delta(v)]}\int_r^{R_\delta(v)} \rho_v(r')dr'$$

which is therefore ≤ 0. Same argument proves the last inequality in (2.13).

□

Lemma 2.7 (Partial order modulo m under diffusion and cut-and-paste)**.** *Let $u \le v$ modulo m, u, v in $\mathcal{U}_\delta$, then*

$$G_t^{\rm neum} * u \le G_t^{\rm neum} * v \quad modulo \quad m \tag{2.14}$$

$$u \le K^{(\delta)} v \quad modulo \quad m + j\delta \tag{2.15}$$

and if $F(0;u) \le F(0;v) + \alpha$ and $m \ge j\delta$, $\alpha \ge 0$, then

$$K^{(\delta)}u \le v \quad modulo \quad \max(m - j\delta, \alpha) \tag{2.16}$$

Proof.

Proof of (2.14). By (2.11) $u^* \le v$ and $F(0; u - u^*) = m$. Then by (2.12)

$$\begin{aligned} F(r; G_t^{\rm neum} * u) &= F(r; G_t^{\rm neum} * u^*) + F(r; G_t^{\rm neum} * (u - u^*)) \\ &\le F(r; G_t^{\rm neum} * v) + F(0; G_t^{\rm neum} * (u - u^*)) = F(r; G_t^{\rm neum} * v) + m \end{aligned}$$

Proof of (2.15). We need to prove that for all $r \ge 0$:

$$F(r;u) \le F(r;K^{(\delta)}v) + m + j\delta$$

For $r = 0$ we have $F(0;v) = F(0;K^{(\delta)}v)$ so that

$$F(0;u) \le F(0;v) + m = F(0;K^{(\delta)}v) + m$$

We next consider $0 < r \le R_\delta(v)$ for which we have

$$F(r;K^{(\delta)}v) = F(r;v) - j\delta \ge F(r;u) - m - j\delta$$

while for $r > R_\delta(v)$ then $F(r;v) \le j\delta$ hence

$$F(r;K^{(\delta)}v) = 0 \ge F(r;v) - j\delta \ge F(r;u) - m - j\delta$$

hence (2.15).

Proof of (2.16). We first observe that for $r = 0$

$$F(0;K^{(\delta)}u) = F(0;u) \le F(0;v) + \alpha$$

As before for $0 < r < R_\delta(u)$,

$$F(r;K^{(\delta)}u) = F(r;u) - j\delta \le F(r;v) + m - j\delta$$

For $r \ge R_\delta(u)$, $F(r;K^{(\delta)}u) = 0 \le F(r;v) \le F(r;v) + m - j\delta$. □

3 Proof of Theorem 1.3

We first state and prove properties of the barriers which will be often used in the sequel. These properties have been proved in [2] when $G_t^{\rm neum}(r,r')$ is replaced by the Green function with Neumann condition both at 0 and at 1. The proofs in [2] extend to our case but the extension is not straightforward because in [2] properties of the particles evolution are sometimes used. We shall give in this paper a self contained proof which is purely analytical and uses only elementary properties of mass transport inequalities.

Proposition 3.1 (Barrier properties)**.** *Let $u, v \in L^\infty(\mathbb{R}_+,\mathbb{R}_+) \cap L^1(\mathbb{R}_+,\mathbb{R}_+)$ and such that $j\delta < F(0;u)$, $j\delta < F(0;v)$. Then*

- *If $u \le v$ then for all $k \in \mathbb{N}$*
$$S_{k\delta}^{(\delta,\pm)}(u) \le S_{k\delta}^{(\delta,\pm)}(v) \tag{3.1}$$

- *Let $\delta = \ell\delta'$ with ℓ a positive integer, then for all $k \in \mathbb{N}$*

$$S_{k\delta}^{(\delta,-)}(u) \le S_{k\delta}^{(\delta',-)}(u) \le S_{k\delta}^{(\delta',+)}(u) \le S_{k\delta}^{(\delta,+)}(u) \tag{3.2}$$

- *For all $k \in \mathbb{N}$*
$$|S_{k\delta}^{(\delta,+)}(u) - S_{k\delta}^{(\delta,-)}(u)|_1 \le 4j\delta \tag{3.3}$$

Proof.

- *Proof of* (3.1)*.* It follows immediately from Lemma 2.6.

- *Proof of* (3.2)*.* It is a consequence of the following inequalities which will be proved next, one after the other:
$$S_{k\delta}^{(\delta,-)}(u) \le S_{k\delta}^{(\delta,+)}(u) \qquad \forall k \in \mathbb{N} \tag{3.4}$$
$$S_{k\delta}^{(\delta',+)}(u) \le S_{k\delta}^{(\delta,+)}(u), \qquad S_{k\delta}^{(\delta,-)}(u) \le S_{k\delta}^{(\delta',-)}(u) \qquad \text{for } \delta = \ell\delta' \quad \forall \ell, k \in \mathbb{N} \tag{3.5}$$

  ⋆ *Proof of* (3.4)*.* We first prove that
$$S_{\delta}^{(\delta,-)}(u) \le S_{\delta}^{(\delta,+)}(u) \tag{3.6}$$
  By (2.15), $u \le K^{(\delta)}u$ modulo $j\delta$, so that using (2.14)
$$G_\delta^{\rm neum} * u \le G_\delta^{\rm neum} * K^{(\delta)}u = S_\delta^{(\delta,+)}(u) \quad \text{modulo } \ j\delta$$
  Then, since $F(0; G_\delta^{\rm neum} * u) = F(0; S_\delta^{(\delta,+)}(u))$, by (2.16) with $\alpha = 0$ and $m = j\delta$ we get
$$S_\delta^{(\delta,-)}(u) = K^{(\delta)} G_\delta^{\rm neum} * u \le S_\delta^{(\delta,+)}(u)$$
  The inequality (3.6) is then proved.

We shall now prove (3.4) by induction on k. The inequality for $k = 1$ follows from (3.6). We next suppose that it holds for k and need to prove it for $k + 1$. Call $u' = S_{k\delta}^{(\delta,-)}(u)$ and $v' = S_{k\delta}^{(\delta,+)}(u)$ then by the induction assumption $u' \le v'$, so that, from (3.1) and (3.6),

$$S_\delta^{(\delta,-)}(u') \le S_\delta^{(\delta,-)}(v') \le S_\delta^{(\delta,+)}(v')$$

hence (3.4) follows because

$$S_{(k+1)\delta}^{(\delta,-)}(u) = S_\delta^{(\delta,-)}(u') \qquad \text{and} \qquad S_{(k+1)\delta}^{(\delta,+)}(u) = S_\delta^{(\delta,+)}(v')$$

⋆ *Proof of* (3.5). We first prove it for $k = 1$:

$$S_\delta^{(\delta',+)}(u) \le S_\delta^{(\delta,+)}(u) \qquad \text{for } \delta = h\delta', \quad h \in \mathbb{N} \tag{3.7}$$

We have

$$S_\delta^{(\delta',+)}(u) = S_{h\delta'}^{(\delta',+)}(u) = G_{\delta'}^{\text{neum}} * K^{(\delta')} \cdots G_{\delta'}^{\text{neum}} * K^{(\delta')} u \qquad h \text{ times}$$

$$S_\delta^{(\delta,+)}(u) = S_{h\delta'}^{(\delta,+)}(u) = G_{\delta'}^{\text{neum}} * \cdots * G_{\delta'}^{\text{neum}} * K^{(\delta)} u \qquad h \text{ times}$$

By applying (2.15) and (2.1) we get

$$u \le K^{(\delta)}(u) \quad \text{modulo } j\delta \qquad \text{and} \quad F(0; u) = F(0; K^{(\delta)} u)$$

then, using (2.16) with $\alpha = 0$ and $m = j\delta$ we get

$$K^{(\delta')} u \le K^{(\delta)} u \quad \text{modulo } \ j\delta - j\delta'$$

then, from (2.14),

$$G_{\delta'}^{\text{neum}} * K^{(\delta')} u \le G_{\delta'}^{\text{neum}} * K^{(\delta)} u \quad \text{modulo } \ j\delta - j\delta'$$

Let now $u' := G_{\delta'}^{\text{neum}} * K^{(\delta')} u$ and $v' := G_{\delta'}^{\text{neum}} * K^{(\delta)} u$ then we have $u' \le v'$ modulo $j\delta - j\delta'$ and $F(0; u') = F(0; v')$. Thus we can apply again (2.16) with $\alpha = 0$ and $m = j\delta - j\delta'$ getting $K^{(\delta')} u' \le v'$ modulo $j\delta - 2j\delta'$. From (2.14) we then get

$$G_{\delta'}^{\text{neum}} * K^{(\delta')} u' \le G_{\delta'}^{\text{neum}} * v' \quad \text{modulo } \ j\delta - 2j\delta'$$

Then we obtain (3.7) by iteration.
We prove now (3.5) by induction on k. We have just proved it for $k = 1$, suppose then that it is verified for k. Call $u' = S_{k\delta}^{(\delta',+)}(u)$, $v' = S_{k\delta}^{(\delta,+)}(u)$ and use (3.7) and the induction assumption $u' \le v'$. Then, from (3.1) and (3.7) we get

$$S_{(k+1)\delta}^{(\delta',+)}(u) = S_\delta^{(\delta',+)}(u') \le S_\delta^{(\delta',+)}(v') \le S_\delta^{(\delta,+)}(v') = S_{(k+1)\delta}^{(\delta,+)}(u)$$

The proof that $S_{k\delta}^{(\delta,-)}(u) \le S_{k\delta}^{(\delta',-)}(u)$ is similar and omitted.

(3.2) is then proved.

- *Proof of* (3.3). Shorthand G for the operator $G_\delta^{\rm neum}*$ and

$$\phi := K^{(\delta)}G\cdots K^{(\delta)}Gu, \qquad \psi := GK^{(\delta)}\cdots GK^{(\delta)}u \qquad k \text{ times}$$

so that we need to bound the total variation of $\phi - \psi$. Call

$$v = K^{(\delta)}u, \qquad v_k = GK^{(\delta)}\cdots Gv, \qquad u_k = GK^{(\delta)}\cdots Gu \qquad k \text{ times}$$

Thus u_k and v_k are obtained by applying $G(K^{(\delta)}G)^{k-1}$ to u and respectively v, hence $\phi = K^{(\delta)}u_k$ and $\psi = v_k$. From (2.2) we have that $G(K^{(\delta)}G)^{k-1}$ is a contraction, then, using (2.3), we get

$$\begin{aligned} |\psi - \phi|_1 &= |K^{(\delta)}u_k - v_k|_1 \le |K^{(\delta)}u_k - u_k|_1 + |v_k - u_k|_1 \\ &= 2j\delta + |v_k - u_k|_1 \le 2j\delta + |u - v|_1 = 4j\delta \end{aligned}$$

Proof of Theorem 1.3. We start by proving the existence of $S_t(u)$. Fix $\tau > 0$ and for any $\ell \in \mathbb{N}$ define $u_{\ell;\tau}(r,t)$ as the linear interpolation of the functions $S_{k2^{-\ell}\tau}^{(2^{-\ell}\tau,+)}(u)$, $k \in \mathbb{N}$. In [2] it is proved that restricted to $t \ge \sigma > 0$ the family $\{u_{\ell;\tau}\}$ is equibounded and equicontinuous. The proof uses only properties of the Green function which are valid both in the domain $[0,1]$ considered in [2] and in our domain $[0,\infty)$. Thus referring to Theorem 2.3 in [2] we can say (by the Ascoli-Arzelà theorem and a diagonalization procedure) that for $t > 0$, $u_{\ell;\tau}(r,t)$ converges by subsequences as $\ell \to \infty$ to a continuous function $u_\tau(r,t)$ which in principle depends also on the converging subsequence. We now prove that for all $r \ge 0$ and $t \in \{k2^{-\ell}\tau;\ \ k,\ell \in \mathbb{N}_+\}$,

$$\lim_{\ell\to\infty} F\Big(r; S_t^{(2^{-\ell}\tau,+)}(u)\Big) = F\big(r; u_\tau(\cdot,t)\big) \tag{3.8}$$

By Fatou's lemma

$$\lim_{\ell\to\infty} F\Big(r; S_t^{(2^{-\ell}\tau,+)}(u)\Big) \ge F\big(r; u_\tau(\cdot,t)\big)$$

so that we just need to prove the converse inequality. Let $R > r$, then

$$F\Big(r; S_t^{(2^{-\ell}\tau,+)}(u)\Big) = \int_r^R S_t^{(2^{-\ell}\tau,+)}(u) + \int_R^\infty S_t^{(2^{-\ell}\tau,+)}(u)$$

The first integral converges to $\displaystyle\int_r^R u_\tau(\cdot,t)$ by the Lebesgue dominated convergence theorem while by (3.2), (2.13) and (2.12)

$$\int_R^\infty S_t^{(2^{-\ell}\tau,+)}(u) \le \int_R^\infty S_t^{(\tau,+)}(u) \le \int_R^\infty G_t^{\rm neum} * u$$

which decays exponentially as $R \to \infty$ (recall that u has compact support). This proves (3.8).

From (3.8) it follows that the limit u_τ is independent of the subsequence, hence that $u_{\ell;\tau}(r,t)$ converges as $\ell \to \infty$ and for $t > 0$ to $u_\tau(r,t)$. In [2], see Theorem 8.4, it is proved that $u_\tau(r,t)$ is actually independent of τ, the proof extends straightforwardly to our case and it is omitted. We can then identify $S_t(u) := u_\tau(\cdot,t)$ and by its definition and (3.2) we know that the equality for the upper barrier in (1.20) is satisfied, i.e. $F(r;S_t(u)) = \lim_{\ell\to\infty} F(r;S_t^{(2^{-\ell}\tau,+)}(u))$. Using (3.3) we get

$$\left|F(r;S_t^{(2^{-\ell}\tau,-)}(u)) - F(r;S_t(u))\right| \le 4j2^{-\ell}\tau + \left|F(r;S_t^{(2^{-\ell}\tau,+)}(u)) - F(r;S_t(u))\right| \tag{3.9}$$

then the equality in (1.20) is satisfied also for the lower barrier. From (3.2) and (1.20) it necessarily follows that

$$F(r;S_t^{(2^{-\ell}\tau,-)}(u)) \le F(r;S_t(u)) \le F(r;S_t^{(2^{-\ell}\tau,+)}(u)) \tag{3.10}$$

this and (1.20) yield (1.21).

The proof that $S_t(u) \to u$ weakly as $t \to 0$ is essentially the same as in [2], Proposition 8.1, and it is omitted. The last statement in the theorem, namely the fact that if $u \le v$ then $S_t(u) \le S_t(v)$, follows directly from (1.20) and (3.1). □

4 Existence of quasi-solutions with arbitrary good accuracy

The analysis in this section, as well as in the following one, exploits extensively a probabilistic representation of the solutions of the heat equation in terms of Brownian motions. We first recall the basic representation formulas and refer to the literature for their proofs. At the end of the section we prove part (a) of Theorem 1.2, namely the existence of optimal quasi-solutions.

4.1 Probabilistic representation of the Green functions

Let $P_{r;s}$, $r \ge 0$, $s \ge 0$ be the law of the Brownian motion B_t, $t \ge s$, which starts from r at time s, i.e. $B_s = r$, and which is reflected at 0. The law of B_t is absolutely continuous with respect to the Lebesgue measure and has a probability density that we denote by $G^{\rm neum}_{s,t}(r,r')$. As a result

$$\rho(r,t) := \int G^{\rm neum}_{s,t}(r',r)\rho(r',s)\,dr', \qquad G^{\rm neum}_{s,t}(r',r)dr = P_{r';s}[B_t \in (r,r+dr)] \tag{4.1}$$

is the solution of the heat equation in $\mathbb{R}_+$ with Neumann conditions at 0 and datum $\rho(r',s)$ at time s; therefore $G^{\rm neum}_{s,t}(r,r')$ is its corresponding Green function.

Call $X = (X_t,\, t \ge 0)$ and denote for $s \ge 0$

$$\tau^X_s = \inf\{t \ge s : B_t \ge X_t\}, \quad \text{and} = \infty \text{ if the set is empty} \tag{4.2}$$

The inf is a minimum as X_t is continuous (see Definition 1.2). The law of B_t, restricted to trajectories so that $\{\tau^X_s > t\}$, has a density with respect to Lebesgue that we denote by $G^{X,\,\rm neum}_{s,t}(r,r')$: for any interval $I \subset \mathbb{R}_+$

$$\int_I G^{X,\,\rm neum}_{s,t}(r',r)dr = P_{r';s}[\tau^X_s > t\,;\, B_t \in I] \tag{4.3}$$

If $\rho(r',s) \in L^\infty([0,X_s),\mathbb{R}_+)$, then the solution of (1.1)–(1.4) (with initial datum $\rho(r',s)$ at time s) is given by

$$\rho(r,t) := \int G^{X,\,\mathrm{neum}}_{s,t}(r',r)\rho(r',s)\,dr' + \int_s^t jG^{X,\,\mathrm{neum}}_{s',t}(0,r)\,ds', \tag{4.4}$$

and therefore $G^{X,\mathrm{neum}}_{s,t}(r,r')$ is its corresponding Green function (recall that X_t is piecewise C^1).

We also have a nice representation for the mass $\Delta^X_{[0,t]}(u)$ which is removed from the system in the time interval $[0,t]$ (due to the Dirichlet boundary conditions when the initial datum is u) in terms of the probability that the Brownian motion reaches the edge X_t.

Lemma 4.1 (Mass loss). *Let $\rho(r,t)$ solve (1.1)–(1.4) with initial datum $u \in L^\infty([0,X_0),\mathbb{R}_+)$ at time 0. Then*

$$F(0,\rho(\cdot,t)) = F(0,u) + jt - \Delta^X_{[0,t]}(u) \tag{4.5}$$

where

$$\Delta^X_{[0,t]}(u) = \int u(r')P_{r';0}\Big[\tau^X_0 \le t\Big]\,dr' + \int_0^t jP_{0;s}\Big[\tau^X_s \le t\Big]\,ds \tag{4.6}$$

In particular mass conservation as in (1.6) requires that $\Delta^X_{[0,t]}(u) = jt$.

Proof. By integration of (4.4), the total mass at any time t is given by

$$F(0,\rho(\cdot,t)) = \int dr'u(r')P_{r',0}[\tau^X_0 > t] + \int_0^t jdsP_{0,s}[\tau^X_s > t] \tag{4.7}$$

Writing $P_{r',0}[\tau^X_0 > t] = 1 - P_{r',0}[\tau^X_0 \le t]$ and $P_{0,s}[\tau^X_s > t] = 1 - P_{0,s}[\tau^X_s \le t]$ one finds

$$F(0,\rho(\cdot,t)) = F(0,u) - \int u(r')P_{r',0}[\tau^X_0 \le t]dr' + jt - \int_0^t jP_{0,s}[\tau^X_s \le t]ds\;. \tag{4.8}$$

Then (4.5) follows from (4.8). □

The distribution of τ^X_0 (inherited from the probability measure $P_{r;0}$) conditioned to the event $\tau^X_0 \le t$ is denoted by $\lambda^X_{r,t}(ds)$. Hence if I is any interval in $\mathbb{R}_+$, $t' \ge t$ then

$$P_{r;0}\Big[B_{t'} \in I \,\Big|\, \tau^X_0 \le t\Big] = \int_0^t P_{X_s,s}\big[B_{t'} \in I]\lambda^X_{r,t}(ds) \tag{4.9}$$

Analogously denoting by $\kappa^X_{s',t}$ the conditional probability density of $\tau^X_{s'}$ given that $\tau^X_{s'} \le t$ (under the probability measure $P_{0;s'}$), we have that that for any $t' \ge t$

$$P_{0;s'}\Big[B_{t'} \in I \,\Big|\, \tau^X_{s'} \le t\Big] = \int_{s'}^t P_{X_s,s}\big[B_{t'} \in I]\kappa^X_{s',t}(ds). \tag{4.10}$$

4.2 Construction of quasi-solutions

The following is a key lemma for the construction of quasi-solutions.

Lemma 4.2. *Let $u \in L^\infty([0,X_0],\mathbb{R}_+)$, $t^*>0$ and $X_t^V = X_0 + Vt$ with $V > V^* := -X_0/t^*$. Call $u^{(V)}(r,t)$ the solution of (1.1)–(1.4) with initial datum u at time 0 and edge X_t^V. Then there is a value of V such that*

$$\Delta^{X^V}_{[0,t^*]}(u) = jt^*, \qquad \sup_{t\le t^*} |\Delta^{X^V}_{[0,t]}(u) - jt| \le jt^* \tag{4.11}$$

Proof. The equality (on the left equation of (4.11)) follows directly from the following statements:

i) $\Delta^{X^V}_{[0,t^*]}(u)$ converges to $jt^* + F(0;u)$ as $V\to V^*$ and it converges to 0 as $V\to\infty$.

ii) $\Delta^{X^V}_{[0,t^*]}(u)$ depends continuously on V in $(V^*,+\infty)$.

The proof of such statements is quite elementary as it can be reduced to simple properties of the Green functions, for completeness we give some details.

i) Let $V>V^*$ and let $\epsilon := X_0 + Vt^* = (V-V^*)t^*$. Then by (4.5) and (4.7)

$$0 \le jt^* + F(0;u) - \Delta^{X^V}_{[0,t^*]}(u) \le \int u(r') P_{r';0}\Big[B_{t^*}\le\epsilon\Big]\,dr' + \int_0^{t^*} jP_{0;s}\Big[B_{t^*}\le\epsilon\Big]\,ds$$

Since B_{t^*} has the law of reflected Brownian motion

$$P_{r';0}\Big[B_{t^*}\le\epsilon\Big] \le \frac{2\epsilon}{\sqrt{2\pi t^*}}, \qquad P_{0;s}\Big[B_{t^*}\le\epsilon\Big] \le \frac{2\epsilon}{\sqrt{2\pi(t^*-s)}}$$

which yields

$$0 \;\le\; jt^* + F(0;u) - \Delta^{X^V}_{[0,t^*]}(u) \le F(0;u)\cdot\frac{2\epsilon}{\sqrt{2\pi t^*}} + \frac{4j\epsilon\sqrt{t^*}}{\sqrt{2\pi}}$$

Hence $\Delta^{X^V}_{[0,t^*]}(u) \to jt^* + F(0;u)$ as $V\to V^*$.

We shall next prove that $\Delta^{X^V}_{[0,t^*]}(u)$ goes to 0 as $V\to\infty$. Let $\epsilon>0$ small, $V=\epsilon^{-\frac34}$ and $r' < X_0 - \epsilon^{\frac14}$. Call $r_k = X_0 + Vt_k$, $t_k = k\epsilon$, then

$$P_{r';0}\Big[\tau_0^{X^V} \le t^*\Big] \le \sum_{k=1}^\infty P_{r';0}\Big[\max_{t\le t_k} B_t \ge r_{k-1}\Big]$$

Denoting by $P^0_{r';0}$ the law of the Brownian motion on the whole $\mathbb{R}$ (i.e. without reflections at 0), we have

$$P_{r';0}\Big[\max_{t\le t_k} B_t \ge r_{k-1}\Big] \le 2P^0_{r';0}\Big[\max_{t\le t_k} B_t \ge r_{k-1}\Big]$$

We then bound

$$P_{r';0}\Big[\max_{t\le t_k} B_t \ge r_{k-1}\Big] \le 4P^0_{r';0}\Big[B_{t_k}\ge r_{k-1}\Big] \le 4e^{-\frac{k}{4\sqrt{\epsilon}}}\int_{k\epsilon^{\frac14}}^\infty \frac{e^{-\frac{x^2}{4k\epsilon}}}{\sqrt{2\pi k\epsilon}} \le 4\sqrt{2}e^{-\frac{k}{4\sqrt{\epsilon}}} \tag{4.12}$$

so that the first term on the right hand side of (4.6) is bounded by

$$\|u\|_1 \epsilon^{\frac{1}{4}} + \|u\|_\infty 4\sqrt{2} \sum_{k=1}^{\infty} e^{-\frac{k}{4\sqrt{\epsilon}}}$$

which vanishes as $\epsilon \to 0$. An analogous argument (which is omitted) applies to the second term on the right hand side of (4.6).

ii) We suppose $V^* < V < V'$ with $(V' - V)t^* =: \epsilon$ and $\epsilon > 0$ small enough. To make notation lighter we shorthand $X = \{X_t = X_0 + Vt\}$ and $X' = \{X'_t = X_0 + V't\}$. Then by (4.6), (4.9) and (4.10),

$$\begin{aligned} \Big|\Delta^X_{[0,t^*]}(u) - \Delta^{X'}_{[0,t^*]}(u)\Big| &\le \int_0^{t^*-\epsilon} g(s)\, P_{X_s;s}\Big[\tau_s^{X'} > t^*\Big]\, ds + R_\epsilon \\ g(s) &= \int dr' u(r') \lambda^X_{r',t^*-\epsilon}(s) + \int_0^s j\kappa^X_{s',t^*-\epsilon}(s) ds' \\ R_\epsilon &:= \int u(r') P_{r';0}\Big[\tau_0^X \in [t^*-\epsilon, t^*]\Big]\, dr' + \int_0^{t^*} j P_{0;s}\Big[\tau_s^X \in [t^*-\epsilon, t^*]\Big]\, ds \end{aligned} \tag{4.13}$$

We are going to prove that there is a function $o(\epsilon)$ which vanishes as $\epsilon \to 0$ so that

$$\sup_{0 \le s \le t^*-\epsilon} P_{X_s;s}\Big[\tau_s^{X'} > t^*\Big] \le o(\epsilon) \tag{4.14}$$

Fix $s \le t^* - \epsilon$ and define $\sigma_s := \inf\{t \ge s \,:\, B_t \notin (X_s - \epsilon^{\frac{3}{4}}, X_s + \alpha\epsilon)\}$, with $\alpha > V' + 1$, then

$$\begin{aligned} P_{X_s;s}\Big[\tau_s^{X'} > t^*\Big] &\le P_{X_s;s}\Big[\sigma_s > s + \epsilon\Big] + P_{X_s;s}\Big[B_{\sigma_s} < X'_{\sigma_s}; \sigma_s \le s + \epsilon\Big] \\ &\le P_{X_s;s}\Big[\sigma_s > s + \epsilon\Big] + P_{X_s;s}\Big[B_{\sigma_s} = X_s - \epsilon^{\frac{3}{4}}\Big] \end{aligned} \tag{4.15}$$

because, by the choice of α, if $B_{\sigma_s} = X_s + \alpha\epsilon$ then $B_{\sigma_s} > X'_{\sigma_s}$ as one can check that $X_s + \alpha\epsilon > X'_{s+\epsilon}$.

Since $P_{r;s}\Big[B_{\sigma_s} = X_s - \epsilon^{\frac{3}{4}}\Big]$ is a linear function of r which has value 1 at $r = X_s - \epsilon^{\frac{3}{4}}$ and is equal to 0 at $r = X_s + \alpha\epsilon$, it then follows that

$$P_{X_s;s}\Big[B_{\sigma_s} = X_s - \epsilon^{\frac{3}{4}}\Big] \le \alpha\, \epsilon^{\frac{1}{4}} \tag{4.16}$$

On the other hand, since the probability density of $B_{s+\epsilon} - X_s$ is $e^{-x^2/(2\epsilon)}(2\pi\epsilon)^{-1/2}$

$$P_{X_s;s}\Big[\sigma_s > s + \epsilon\Big] \le P_{X_s;s}\Big[|B_{s+\epsilon} - X_s| \le \epsilon^{\frac{3}{4}}\Big] \le \frac{2}{\sqrt{2\pi}} \cdot \epsilon^{\frac{1}{4}} \tag{4.17}$$

so that (4.14) is proved. We then have that the first term on the right hand side of (4.13) is bounded by:

$$o(\epsilon) \int_0^{t^*} g(s)\, ds \le o(\epsilon) \cdot (F(0; u) + jt^*)$$

We shall next bound the probabilities in R_ϵ. Call $Y = X_{t^*-\epsilon} = X_0 + V(t^* - \epsilon)$, then

$$P_{r';0}\Big[\tau_0^X \in [t^*-\epsilon, t^*]\Big] \le P_{r';0}\Big[B_{t^*-\epsilon} \in [Y-\epsilon^{\frac{1}{4}}, Y]\Big] + \sup_{r'' \le Y-\epsilon^{\frac{1}{4}}} P_{r'';t^*-\epsilon}\Big[\max_{t\in[t^*-\epsilon,t^*]} B_t \ge Y\Big] \tag{4.18}$$

As before we have

$$P_{r';0}\Big[B_{t^*-\epsilon} \in [Y-\epsilon^{\frac{1}{4}}, Y]\Big] \le \frac{\epsilon^{\frac{1}{4}}}{\sqrt{2\pi(t^*-\epsilon)}}$$

Now suppose $r'' \in [0, Y - \epsilon^{\frac{1}{4}}]$, then

$$P_{r'';t^*-\epsilon}\Big[\max_{t\in[t^*-\epsilon,t^*]} B_t \ge Y\Big] \le P_{r'';t^*-\epsilon}\Big[\max_{t\in[t^*-\epsilon,t^*]} (B_t - r'') \ge \epsilon^{\frac{1}{4}}\Big]$$

By the same argument used in (4.12), the latter is bounded by

$$2P_{r';t^*-\epsilon}\Big[B_{t^*} - r' \ge \epsilon^{\frac{1}{4}}\Big] \le 2\int_{\epsilon^{\frac{1}{4}}}^{\infty} \frac{e^{-\frac{x^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}}\, dx \le 4\sqrt{2}e^{-\frac{1}{4\sqrt{\epsilon}}}$$

Analogous bounds are proved for $P_{0;s}\Big[\tau_s^X \in [t^*-\epsilon, t^*]\Big]$, we omit the details. We have thus proved that also R_ϵ is infinitesimal with ϵ.

We have thus proved the identity in (4.11). The second statement (i.e. the inequality in (4.11)) follows because $\Delta^{XV}_{[0,t]}(u)$ is a non-negative, non decreasing function of t then its maximum in $t \in [0, t^*]$ is $\Delta^{XV}_{[0,t^*]}(u) = jt^*$. □

Proof of Theorem 1.2, part (a). For each positive integer n consider the time grid of mesh ϵ_n and given $\rho^{(n)}(r, 0)$ use the above lemma to construct $\rho^{(n)}(r, t)$ in the first interval of the grid, $t \le t^* = \epsilon_n$ observing that at the final time the total mass is exactly equal to the initial one. We can then use again the lemma starting from $\rho^{(n)}(r, \epsilon_n)$ to construct $\rho^{(n)}(r, t)$ in the second interval, $t \in [\epsilon_n, 2 \cdot \epsilon_n]$, at the end of which we still have conservation of the total mass. By iteration $\rho^{(n)}(r, t)$ is then defined for all $t \in [0, T]$, mass is conserved at all discrete times of the grid and the mass conservation can be violated only in the interior of the intervals of the time grid. By Lemma 4.2, at any time $t \in [0, T]$ we have conservation of mass modulo at most $j\epsilon_n$. □

5 Characterization and uniqueness

In this section we characterize the function $\bar\rho$ of Theorem 1.2 as the unique separating element between barriers thus proving the existence and uniqueness of the generalized solution.

5.1 The key inequality

We fix a function ρ_0, $T > 0$ and a quasi-solution $(X_t, \rho(\cdot, t), \epsilon)$, $t \in [0, T]$ with accuracy parameter ϵ as in Definition 1.2. Recalling Definition 2.1 we prove the following key inequality.

Proposition 5.1. *For any $\delta > 0$, there is a constant c so that for all $k \in \mathbb{N}$ such that $k\delta \le T$*

$$S_{k\delta}^{(\delta,-)}(\rho(\cdot,0)) \le \rho(\cdot,k\delta) \le S_{k\delta}^{(\delta,+)}(\rho(\cdot,0)) \qquad \textit{modulo} \quad ck\epsilon \tag{5.1}$$

Proof. We prove (5.1) by induction on k. We thus fix k, shorthand

$$v^{(\pm)} := S_{k\delta}^{(\delta,\pm)}(\rho(\cdot,0)), \qquad w = \rho(\cdot,k\delta), \qquad Z_t := X_{k\delta+t}, \qquad m = ck\epsilon \tag{5.2}$$

and suppose (by induction) that

$$v^{(-)} \le w \ \text{ modulo } m \qquad\qquad \text{and} \qquad\qquad w \le v^{(+)} \ \text{ modulo } m \tag{5.3}$$

We call $w(r,t) = \rho(r, k\delta + t)$, $t \in [0,\delta]$. We shall prove in Subsection 5.2 that

$$S_{\delta}^{(\delta,-)}(v^{(-)}) \le w(\cdot,\delta) \qquad \text{modulo } \ m + c\epsilon \tag{5.4}$$

and in Subsection 5.3 that

$$w(\cdot,\delta) \le S_{\delta}^{(\delta,+)}(v^{(+)}) \qquad \text{modulo } \ m + c\epsilon \tag{5.5}$$

which will thus prove the induction and concludes the proof of the Proposition. □

We conclude this subsection by observing two properties of Z_t and $w(\cdot,t)$ (see (5.2)) that, together with (1.11), will be extensively used in next two subsections for the proofs of the upper and lower bound.

$(Z_t, w(\cdot,t), 2\epsilon)$, $t \in [0,\delta]$, is a quasi-solution of the FBP 1.1 with initial datum w and accuracy parameter 2ϵ (see (1.11)), then from (4.4) and (4.3) it follows that for any $I \subseteq \mathbb{R}^+$,

$$\int_I w(r,t)\,dr = \int w(r')P_{r';0}\Big[\tau_0^Z > t; B_t \in I\Big]\,dr' + \int_0^t jP_{0;s}\Big[\tau_s^Z > t; B_t \in I\Big]\,ds \tag{5.6}$$

Moreover if $\Delta_{[0,t]}^Z(w)$ is the mass lost by w in the time interval $[0,t]$, $0 \le t \le \delta$ (see (4.6)), then from (4.5) and (1.11) we have

$$\sup_{t\in[0,\delta]} \Big|\Delta_{[0,t]}^Z(w) - jt\Big| \le 2\epsilon\ . \tag{5.7}$$

5.2 Lower bound (proof of (5.4)).

We write here v for $v^{(-)}$ and by the induction hypothesis we have that $v \le w$ modulo m. We need to prove that for all $r \ge 0$

$$F(r; K^{(\delta)}G_\delta^{\text{neum}} * v) \le F(r; w(\cdot,\delta)) + m + c\epsilon \tag{5.8}$$

We decompose $v = v^* + v_1$ with

$$v^* = v\,\mathbf{1}_{[0,R_m]}, \qquad R_m : \int_{R_m}^\infty v = m, \qquad v_1 = v\,\mathbf{1}_{(R_m,+\infty]}$$

so that, by Lemma 2.5, $v^* \le w$. Let

$$w = \tilde w + w_1, \qquad w_1 = w\mathbf{1}_{[0,\tilde R]}, \qquad\qquad \tilde R : \int w_1 = F(0;w) - F(0;v^*) \ge 0$$

(because $v^* \le w$), so that by Lemma 2.4,

$$v^* \le \tilde w, \qquad F(0;v^*) = F(0;\tilde w)$$

From the right identity in (4.1) we have

$$F(r; G_\delta^{\rm neum} * v^*) = \int dr' v^*(r') \int_r^\infty G_\delta^{\rm neum}(r',z)\, dz = \int \ v^*(r')\ P_{r';0}\Big[B_\delta \ge r\Big] dr' \tag{5.9}$$

We define $\tilde w(r,t)$, $t \in [0,\delta]$, the evolution at time t of the profile $\tilde w(r)$ with the dynamics defined by (5.6), then

$$F(r;\tilde w(\cdot,\delta)) = \int \ \tilde w(r')\, P_{r';0}\Big[\tau_0^Z > \delta;\, B_\delta \ge r\Big] dr' + \int_0^\delta j\, P_{0;s}\Big[\tau_s^Z > \delta;\, B_\delta \ge r\Big] ds \tag{5.10}$$

Let $\Delta^Z_{[0,\delta]}(w)$ and $\Delta^Z_{[0,\delta]}(\tilde w)$ be the mass lost by w and $\tilde w$ in the time interval $[0,\delta]$. Since $\tilde w(r) \le w(r)$ for all r, then, from (4.6) it follows that $\Delta^Z_{[0,\delta]}(\tilde w) \le \Delta^Z_{[0,\delta]}(w)$, then, from (5.7) we have

$$\Delta^Z_{[0,\delta]}(\tilde w) \le \Delta^Z_{[0,\delta]}(w) \le j\delta + 2\epsilon \tag{5.11}$$

From (4.6) and (5.10) we get

$$F(r;\tilde w(\cdot,\delta)) \quad\ge\quad \int \ \tilde w(r')\, P_{r';0}\Big[B_\delta \ge r\Big] dr' + \int_0^\delta jP_{0;s}\Big[B_\delta \ge r\Big] ds - \Delta^Z_{[0,\delta]}(\tilde w) \tag{5.12}$$

The function $\phi(r') := P_{r';0}\Big[B_\delta \ge r\Big]$ is bounded and non decreasing, then from (2.8)

$$\int \ v^*(r')\, \phi(r') dr' \le \int \ \tilde w(r')\, \phi(r') dr' \tag{5.13}$$

hence

$$\begin{aligned} F(r;\tilde w(\cdot,\delta)) &\ge \int v^*(r') P_{r';0}\Big[B_\delta \ge r\Big] dr' + \int_0^\delta j\, P_{0;s}\Big[B_\delta \ge r\Big] ds - \Delta^Z_{[0,\delta]}(\tilde w) \\ &\ge F(r; G_\delta^{\rm neum} * v^*) + \int_0^\delta j\, P_{0;s}\Big[B_\delta \ge r\Big] ds - (j\delta + 2\epsilon) \end{aligned}$$

where the last inequality follows from (5.9) and (5.11). Then

$$\begin{aligned} &\tilde w(\cdot,\delta) \ge G_\delta^{\rm neum} * v^* \qquad \text{modulo } \ j\delta + 2\epsilon \\ \text{and} \qquad &F(0;\tilde w(\cdot,\delta)) \ge F(0; G_\delta^{\rm neum} * v^*) - 2\epsilon \end{aligned}$$

Using (2.16) with $m = j\delta + 2\epsilon$ and $\alpha = 2\epsilon$ we get

$$\tilde w(\cdot,\delta) \ge K^{(\delta)} G_\delta^{\rm neum} * v^* \qquad \text{modulo } \ 2\epsilon$$

On the other hand

$$K^{(\delta)} G_\delta^{\rm neum} * v^* \ge K^{(\delta)} G_\delta^{\rm neum} * v \qquad \text{modulo } \ F(0; K^{(\delta)} G_\delta^{\rm neum} * v_1)$$

where $F(0; K^{(\delta)} G_\delta^{\rm neum} * v_1) = F(0; v_1) = m$ so that

$$\tilde w(\cdot,\delta) \ge K^{(\delta)} G_\delta^{\rm neum} * v \qquad \text{modulo } \ m + 2\epsilon$$

which proves (5.8) (with $c \ge 1$) because $w(\cdot,\delta) \ge \tilde w(\cdot,\delta)$. □

5.3 Upper bound (proof of (5.5)).

We shorthand here v for $v^{(+)}$, then, by the induction assumption, $w \le v$ modulo m where $w(r,t)$, $t \in [0,\delta]$, is given by (5.6). We have to prove that

$$I(r) := F(r; w(\cdot,\delta)) - F(r; G_\delta^{\rm neum} * K^{(\delta)} v) - m \le c\epsilon, \quad \text{for all } r \ge 0 \tag{5.14}$$

We write $w = w_0 + w_1 + w_2$ with

$$w_2 = w\,\mathbf{1}_{(R_m(w),+\infty)} : \quad \int w_2 = m; \qquad\qquad w_1 = w\,\mathbf{1}_{[R_1,R_m(w)]} : \quad \int w_1 = j\delta \tag{5.15}$$

Next lemma shows that after some simple manipulations the proof of the upper bound is reduced to some inequalities among the w_i.

Lemma 5.2. *Let $I(r)$ be the function defined in (5.14), then*

$$I(r) \le \int w_1(r') P_{r';0}\Big[B_\delta \ge r; \tau_0^Z > \delta\Big] dr' - \sum_{i\in\{0,2\}} \int w_i(r') P_{r';0}\Big[B_\delta \ge r; \tau_0^Z \le \delta\Big] dr'$$
$$-j \int_0^\delta P_{0;s}\Big[B_\delta \ge r; \tau_s^Z \le \delta\Big] ds \tag{5.16}$$

Proof. Since $w \le v$ modulo m, by (2.11) (and the definition of w_2) $w_0 + w_1 \le v$. We write $v = v_0 + v_1 + v_2$ where

$$v_1 = v\,\mathbf{1}_{[R_\delta(v),+\infty)} : \quad \int v_1 = j\delta \tag{5.17}$$

$$v_2 = v\,\mathbf{1}_{[0,\tilde R]} : \quad \int v_2 = F(0;v) - F(0; w_0 + w_1)$$

so that by (2.9)

$$w_0 + w_1 \le v_0 + v_1, \quad F(0; w_0 + w_1) = F(0; v_0 + v_1)$$

then

$$F(0; v_i) = F(0; w_i), \qquad i = 0, 1$$

hence, by the last inequality in (2.13),

$$w_0 \le v_0, \qquad F(0; v_0) = F(0; w_0) \tag{5.18}$$

Observe moreover that

$$F(r; S_\delta^{(\delta,+)}(v_0 + v_1)) \le F(r; S_\delta^{(\delta,+)}(v)), \qquad S_\delta^{(\delta,+)}(v_0 + v_1) = G_\delta^{\rm neum} * [v_0 + j\delta D_0] \tag{5.19}$$

We call

$$f_i(r) = \int w_i(r')\, P_{r';0}\Big[B_\delta \ge r;\ \tau_0^Z > \delta\Big] dr', \qquad i = 0, 1, 2 \tag{5.20}$$

$$f_3(r) = j \int_0^\delta P_{0;s}\Big[B_\delta \ge r;\ \tau_s^Z > \delta\Big] ds$$

so that, from (5.6) we have

$$F(r; w(\cdot, \delta)) = \sum_{i=0}^{3} f_i(r) \tag{5.21}$$

For $i \in \{0, 1, 2\}$ we write

$$f_i(r) = \int w_i(r') P_{r';0}\Big[B_\delta \geq r\Big] dr' - \int w_i(r') P_{r';0}\Big[B_\delta \geq r; \tau_0^Z < \delta\Big] dr' \tag{5.22}$$

and for $i = 3$:

$$f_3(r) = j \int_0^\delta P_{0;s}\Big[B_\delta \geq r\Big] ds - j \int_0^\delta P_{0;s}\Big[B_\delta \geq r;\ \tau_s^Z < \delta\Big] ds \tag{5.23}$$

Using (5.21) and the left inequality in (5.19) we then get

$$I(r) \leq \sum_{i=0}^{3} f_i(r) - F(r; S_\delta^{(\delta,+)}(v_0 + v_1)) - m \tag{5.24}$$

with $f_i(r)$ as in (5.22)–(5.23) when $i \neq 1$ and by $f_1(r)$ as in (5.20). From the identity in (5.18) and (4.1) we get

$$F(r; S_\delta^{(\delta,+)}(v_0 + v_1)) = F\Big(r; G_\delta^{\mathrm{neum}} * [v_0(\cdot, \delta) + j\delta D_0]\Big) = \int v_0(r') P_{r';0}\Big[B_\delta \geq r\Big] dr' + j\delta P_{0;0}\Big[B_\delta \geq r\Big] \tag{5.25}$$

From (2.12) and (4.1) we have

$$\int w_0(r') P_{r';0}\Big[B_\delta \geq r\Big] dr' \leq \int v_0(r') P_{r';0}\Big[B_\delta \geq r\Big] dr' \tag{5.26}$$

moreover

$$j \int_0^\delta P_{0;s}\Big[B_\delta \geq r\Big] ds \leq j\delta P_{0;0}\Big[B_\delta \geq r\Big]$$

$$\text{and} \qquad \int w_2(r') P_{r';0}\Big[B_\delta \geq r\Big] dr' \leq m \tag{5.27}$$

then (5.16) follows from (5.24), (5.25), (5.26) and (5.27). □

From the above lemma we are left with the proof that

$$\int w_1(r') P_{r';0}\Big[B_\delta \geq r;\ \tau_0^Z > \delta\Big] dr' \leq \sum_{i \in \{0,2\}} \int w_i(r') P_{r';0}\Big[B_\delta \geq r;\ \tau_0^Z \leq \delta\Big] dr'$$
$$+ j \int_0^\delta P_{0;s}\Big[B_\delta \geq r; \tau_s^Z \leq \delta\Big] ds + c\epsilon \tag{5.28}$$

Call

$$\alpha(r) := P_{r;0}\Big[\tau_0^Z \leq \delta\Big], \qquad \beta(s) := P_{0;s}\Big[\tau_s^Z \leq \delta\Big] \tag{5.29}$$

Then the inequality in (5.28) becomes

$$\int w_1(r')[1-\alpha(r')]P_{r';0}\Big[B_\delta \ge r \,\Big|\, \tau_0^Z > \delta\Big]dr' \le \sum_{i=0,2}\int w_i(r')\alpha(r')P_{r';0}\Big[B_\delta \ge r \,\Big|\, \tau_0^Z \le \delta\Big]dr'$$
$$+j\int_0^\delta \beta(s)P_{0;s}\Big[B_\delta \ge r \,\Big|\, \tau_s^Z \le \delta\Big]ds + c\epsilon \tag{5.30}$$

By using (4.5) and (5.7), we get

$$\big|F(0,w(\cdot,\delta)) - F(0,w)\big| = \big|\Delta^Z_{[0,\delta]}(w) - j\delta\big| \le 2\epsilon \tag{5.31}$$

From (5.6) we have

$$F(0,w(\cdot,\delta)) - F(0,w) = j\delta - \sum_{i=0}^{2}\int \alpha(r)w_i(r)dr - j\int_0^\delta \beta(s)ds$$

Since $\int w_1 = j\delta$ we have

$$\Big|\Big\{\sum_{i=0}^{2}\int \alpha(r)w_i(r)dr + j\int_0^\delta \beta(s)ds\Big\} - \int w_1(r)dr\Big| \le 2\epsilon \tag{5.32}$$

Since (5.30) is trivially satisfied for any $c \ge 2$ when $\int w_1(r)[1-\alpha(r)]\,dr \le 2\epsilon$, we can suppose that $\int w_1(r)[1-\alpha(r)]\,dr > 2\epsilon$. It follows that there is an interval $I \in \mathbb{R}_+$ so that

$$\int_I w_1(r)[1-\alpha(r)]\,dr = 2\epsilon$$

By (5.32) there exists $q \le 1$ so that

$$M := \int_{I^c} w_1(r)[1-\alpha(r)] = q\Big\{\sum_{i=0,2}\int \alpha(r)w_i(r) + j\int_0^\delta \beta(s)\Big\} \tag{5.33}$$

We are going to prove that there exists a constant c' so that

$$\int_{I^c} w_1(r')[1-\alpha(r')]P_{r';0}\Big[B_\delta \ge r \,\Big|\, \tau_0^Z > \delta\Big]dr' \le q\Big(\sum_{i=0,2}\int w_i(r')\alpha(r')P_{r';0}\Big[B_\delta \ge r \,\Big|\, \tau_0^Z \le \delta\Big]dr'$$
$$+j\int_0^\delta \beta(s)P_{0;s}\Big[B_\delta \ge r \,\Big|\, \tau_s^Z \le \delta\Big]\Big)ds + c'\epsilon \tag{5.34}$$

which yields (5.30) with $c = c' + 2$.

Recalling (4.9)–(4.10) there exists a non negative measure $g(dt)$ on $[0,\delta]$, so that $\int_0^\delta g(dt) = M$ and

$$q\Big(\sum_{i=0,2}\int dr' w_i(r')\alpha(r')P_{r';0}\Big[B_\delta \ge r \,\Big|\, \tau_0^Z \le \delta\Big]$$
$$+j\int_0^\delta ds\beta(s)P_{0;s}\Big[B_\delta \ge r \,\Big|\, \tau_s^Z \le \delta\Big]\Big) = \int_0^\delta g(dt)P_{Z_t;t}\Big[B_\delta \ge r\Big]$$

Since $w_1(r)[1-\alpha(r)]\mathbf{1}_{I^c}(r)dr =: \mu(dr)$ and $g(dt)$ have same mass M, and $g(dt)$ does not have atomic components, then, by the isomorphism of Lebesgue measures, [29], there is a map $\Gamma : \mathbb{R}_+ \to [0,\delta]$ so that

$$\int_0^\delta g(dt) P_{Z_t;t}\Big[B_\delta \geq r\Big] = \int \mu(dr') P_{Z_{\Gamma(r')};\Gamma(r')}\Big[B_\delta \geq r\Big] \tag{5.35}$$

(5.34) with $c' = 0$ will then follow from the inequality

$$P_{r';0}\Big[B_\delta \geq r \,\big|\, \tau_0^Z > \delta\Big] \leq P_{Z_t;t}\Big[B_\delta \geq r\Big], \quad r' \in [0, Z_0),\ t \in [0,\delta) \tag{5.36}$$

(used with $t = \Gamma(r')$). A proof of (5.36) via a coupling between conditioned and unconditioned Brownian motions which preserves order is given in [6]. Here we present a more elementary proof where we use classical couplings between Brownian motions (reflected at the origin).

Proposition 5.3. *Let $P_{r,r',s} = P_{r,s} \times P_{r',s}$, $r, r' \in \mathbb{R}_+$, be the product of two independent standard Brownians, $B_t^{(1)}$ and $B_t^{(2)}$, $t \geq s$, on $\mathbb{R}_+$ with reflections at* 0*. Call $\tau := \inf\{s : B_s^{(1)} = B_s^{(2)}\}$ (and equal to $+\infty$ if the set is empty). Then*

$$b_t := \begin{cases} B_t^{(2)} & \text{if } t \leq \tau \\ B_t^{(1)} & \text{if } t \geq \tau \end{cases} \tag{5.37}$$

has the law of a Brownian motion starting from r' at time s.

Proof of (5.36)**.** We use the method of duplicating variables. Let γ^{-1} be a positive integer (eventually $\gamma \to 0$), $B_i^{(1)}$, $i = 1, .., \gamma^{-1}$ independent Brownian motions which start moving at time t from Z_t (and are frozen before: $B_i^{(1)}(s) = Z_t$, $s \leq t$). Then denoting below by $E^{(1)}$ the expectation with respect to such independent Brownians,

$$P_{Z_t;t}\Big[B_\delta \geq r\Big] = E^{(1)}\Big[\gamma \sum_i \mathbf{1}_{[r,+\infty)}(B_i^{(1)}(\delta))\Big] \tag{5.38}$$

We can proceed in analogous way with $P_{r';0}\Big[B_\delta \geq r \,\big|\, \tau_0^Z > \delta\Big]$ which is now conveniently rewritten as

$$P_{r';0}\Big[B_\delta \geq r \,\big|\, \tau_0^Z > \delta\Big] = P_{r';0}\Big[B_\delta \geq r\,;\, \tau_0^Z > \delta\Big]\Big(1 + \Big\{\frac{1}{1-\alpha(r')} - 1\Big\}\Big) \tag{5.39}$$

Calling $N_\gamma :=$ the integer part of $\gamma^{-1}\big\{\frac{1}{1-\alpha(r')} - 1\big\}$, we then consider $B_i^{(2)}$, $i = 1, .., \gamma^{-1} + N_\gamma$ independent Brownian motions which start at time 0 from r' and are removed once they reach the edge Z_t. Then, denoting below by $E^{(2)}$ expectation with respect to the law of the independent Brownians with death at the edge,

$$P_{r';0}\Big[B_\delta \geq r \,\big|\, \tau_0^Z > \delta\Big] = \lim_{\gamma \to 0} E^{(2)}\Big[\gamma \sum_{i=1}^{\gamma^{-1}+N_\gamma} \mathbf{1}_{[r,+\infty)}(B_i^{(2)}(\delta))\Big] \tag{5.40}$$

The equality holds only in the limit because of the integer part in the definition of N_γ.

Call $i_1,..,i_{M_\gamma}$ the labels of the $B^{(2)}$-particles still existing at time t. To simplify notation we relabel them as $1,..,M_\gamma$ and denote by $r_1,..,r_{M_\gamma}$ the positions of the corresponding $B^{(2)}$-particles at time t. Thus at time t we have γ^{-1} $B^{(1)}$-particles all at Z_t and a number M_γ of $B^{(2)}$-particles at r_1,..,r_{M_γ} (all $< Z_t$, otherwise they would be dead).

We call married couples at time t the pairs $(B_i^{(1)}(t), B_i^{(2)}(t))$ with $i \le \min\{\gamma^{-1}, M_\gamma\}$, single the $B_i^{(2)}(t)$ particles with $i > \gamma^{-1}$, if any. We then let evolve all the particles present at time t in the following way: each married couple moves as in Proposition 5.3 independently of the other couples and of all the other particles which move as independent Brownians. At the first time $s > t$ when a $B_i^{(2)}$ particle reaches the edge Z_s, it disappears. If it was single then the evolution after s proceeds as before without the dead particle. If instead the particle which reaches the edge at time $s > t$ is one of the "married particles" the $B^{(1)}$-particle in the pair is coupled with a single $B^{(2)}$-particle, if such a particle exists at time s, otherwise it is an unmarried $B^{(1)}$-particle. After that the process continues with same rules and it is thus defined completely by iteration. We denote by P and E law and expectation of the coupled process.

By its definition it follows that the number K_γ of single $B^{(2)}$-particles at the final time δ is

$$K_\gamma = \max\Big\{0; N_\gamma - \sum_{i=1}^{\gamma^{-1}+N_\gamma} \mathbf{1}_{B_i^{(2)}(s)=Z_s,\ \text{for some } s\,\in\,[0,\delta]}\Big\}$$

while for all pairs $(B_i^{(1)}(\delta), B_i^{(2)}(\delta))$ present at time δ the inequalities $B_i^{(1)}(\delta) \ge B_i^{(2)}(\delta)$ hold. Thus by (5.38)–(5.40)

$$P_{Z_t;t}\Big[B_\delta \ge r\Big] - P_{r';0}\Big[B_\delta \ge r \,\big|\, \tau_0^Z > \delta\Big] \ge -\lim_{\gamma\to 0} E\Big[\gamma K_\gamma\Big] \tag{5.41}$$

By the law of large numbers for independent variables, for any $\zeta > 0$

$$\lim_{\gamma\to 0} P_{r',0}\Big[\,\Big|\gamma \sum_{i=1}^{\gamma^{-1}+N_\gamma} \mathbf{1}_{B_i^{(2)}(s)=Z_s,\ \text{for some } s\,\in\,[0,\delta]} - (1+\gamma N_\gamma) P_{r',0}[\tau_0^Z \le \delta]\Big| \le \zeta\Big] = 1 \tag{5.42}$$

Since

$$\lim_{\gamma\to 0}(1+\gamma N_\gamma) P_{r',0}[\tau_0^Z \le \delta] = \frac{\alpha(r')}{1-\alpha(r')} = \lim_{\gamma\to 0} \gamma N_\gamma$$

it follows from (5.42) that

$$\lim_{\gamma\to 0} P_{r',0}\Big[\,\Big|\gamma \sum_{i=1}^{\gamma^{-1}+N_\gamma} \mathbf{1}_{B_i^{(2)}(s)=Z_s,\ \text{for some } s\,\in\,[0,\delta]} - \gamma N_\gamma\Big| \le \zeta\Big] = 1$$

hence $\lim_{\gamma\to 0} P\big[\gamma K_\gamma \le \zeta\big] = 1$ which yields $\lim_{\gamma\to 0} E\big[\gamma K_\gamma\big] = 0$, thus the right hand side of (5.41) is equal to 0.

□

5.4 Conclusion of the proof of Theorem 1.2 and Theorem 1.4

In all this subsection $(X_t^{(n)}, \rho^{(n)}, \epsilon_n)$, $t \in (0,T)$ is an optimal sequence of quasi-solutions as in part (b) of Theorem 1.2. The following result is an easy consequence of Proposition 5.1.

Corollary 5.4. *For any fixed $t \in (0,T)$ the sequence of non negative measures $\{\rho^{(n)}(r,t)dr\}$ on $\mathbb{R}_+$ with its Borel σ-algebra is tight (in the weak topology of measures).*

Proof. Fix arbitrarily $t \in (0,T)$ and $\ell \in \mathbb{N}$, call $\delta = 2^{-\ell}t$ and $k^* = 2^\ell$ so that $k^*\delta = t$. Then from (5.1) with such value of δ and with $k = k^*$, we get that there exists a constant c so that for any n

$$S_t^{(\delta,-)}(\rho^{(n)}(\cdot,0)) \le \rho^{(n)}(\cdot,t) \le S_t^{(\delta,+)}(\rho^{(n)}(\cdot,0)) \qquad \text{modulo} \quad ck^*\epsilon_n \tag{5.43}$$

To prove tightness we use the Prokhorov theorem: it is then sufficient to show that for any $\zeta > 0$ there is r_ζ so that

$$F(r_\zeta;\rho^{(n)}(\cdot,t)) \le \zeta \quad \text{for all } n \tag{5.44}$$

From (5.43) it follows that

$$F(r;\rho^{(n)}(\cdot,t)) \le F(r;S_t^{(\delta,+)}(\rho^{(n)}(\cdot,0))) + ck^*\epsilon_n \tag{5.45}$$

By (2.4) and the definition of quasi-solution

$$\begin{aligned}\big|F(r;S_t^{(\delta,+)}(\rho^{(n)}(\cdot,0))) - F(r;S_t^{(\delta,+)}(\rho_0))\big| &\le \int \big|S_t^{(\delta,+)}(\rho^{(n)}(\cdot,0)) - S_t^{(\delta,\pm)}(\rho_0)\big|(r)\,dr \\ &\le \int \big|\rho^{(n)}(r,0) - \rho_0(r)\big|\,dr \le \epsilon_n \end{aligned}\tag{5.46}$$

Moreover for any ζ there is R so large that for all $r \ge R$

$$F(r;S_t^{(\delta,+)}(\rho_0))) \le 2jt\int_r^\infty \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}dx + \int_r^\infty dx\int_{\mathbb{R}_+} dr'\rho_0(r')G^{\text{neum}}(r',x) \le \frac{\zeta}{2} \tag{5.47}$$

Given ζ let n_0 be so that for all $n \ge n_0$, $\epsilon_n(1+c\delta^{-1}T) \le \zeta/2$, then (5.44) follows from (5.45), (5.46) and (5.47) and thus the corollary is proved. □

Proof of (b) and (c) of Theorem 1.2. From Corollary 5.4 the sequence $\{\rho^{(n)}(r,t)dr\}_{n\ge 0}$ converges weakly by subsequences to a limit measure $\mu(dr)$ and we next prove that $\mu(dr) = S_t(\rho_0)dr$. This implies that the sequence itself converges and it identifies the function $\bar\rho$ in (c), thus concluding both the proof of Theorem 1.2 and the first statement of Theorem 1.4.

Call

$$f'_{r,t} := \liminf_{n\to\infty} F(r;\rho^{(n)}(\cdot,t)), \quad f''_{r,t} := \limsup_{n\to\infty} F(r;\rho^{(n)}(\cdot,t)) \tag{5.48}$$

Let again $\delta = 2^{-\ell}t$, $\ell \in \mathbb{N}$, then by (5.43)

$$F(r;S_t^{(\delta,-)}(\rho^{(n)}(\cdot,0))) - c\epsilon_n\delta^{-1}T \le F(r;\rho^{(n)}(\cdot,t)) \le F(r;S_t^{(\delta,+)}(\rho^{(n)}(\cdot,0))) + c\epsilon_n\delta^{-1}T$$

By (5.46)

$$F(r;S_t^{(\delta,-)}(\rho_0)) - \zeta_n \le F(r;\rho^{(n)}(\cdot,t)) \le F(r;S_t^{(\delta,+)}(\rho_0)) + \zeta_n \tag{5.49}$$

where $\zeta_n := \epsilon_n(c\delta^{-1}T + 1)$. Since $\zeta_n \to 0$ as $n \to \infty$ and recalling that $\delta = 2^{-\ell}t$:

$$F(r;S_t^{(2^{-\ell}t,-)}(\rho_0)) \le f'_{r,t} \le f''_{r,t} \le F(r;S_t^{(2^{-\ell}t,+)}(\rho_0)) \tag{5.50}$$

By (1.20) letting $\ell \to \infty$

$$f'_{r,t} = f''_{r,t} = F(r; S_t(\rho_0)) \tag{5.51}$$

which, by the arbitrariness of r, identifies the limit measure $\mu(dr) = S_t(\rho_0)(r)dr$. □

Proof of Theorem 1.4. The first statement has been already proved above. The function $\rho(r,t) := \rho_a(r)$, see (1.7), is a classical (and stationary) solution in the sense of Definition 1.1, hence it is a fortiori a quasi-solution (with $X_t \equiv \frac{a}{2j}$ and accuracy parameter $\epsilon = 0$). Then for what proved so far, $S_t(\rho_a) = \rho_a$. Let ρ_0 be as in Theorem 1.4, then, since $R(\rho_0) < \infty$ for a large enough $\rho_0(r) \le \rho_a(r)$ for all r hence $\rho_0 \le \rho_a$ also in the sense of mass transport. Since S_t preserves order (see the last statement in Theorem 1.3), we have $S_t(\rho_0) \le S_t(\rho_a) = \rho_a$ which proves that $R(\rho(\cdot,t)) \le R(\rho_a) = \frac{a}{2j} := a_2$ for all $t \ge 0$. Analogously, since $R(\rho_0) > 0$ for $\bar{a}$ such that $\frac{\bar{a}}{2j} < R(\rho_0)$ we get that $\rho_{\bar{a}} = S_t(\rho_{\bar{a}}) \le S_t(\rho_0)$ which proves that $R(\rho(\cdot,t)) \ge R(\rho_{\bar{a}}) = \frac{\bar{a}}{2j} := a_1$ for all $t \ge 0$. □

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References

[1] J. Anselmi, B. D'Auria, N. Walton, Closed queueing networks under congestion: non-bottleneck independence and bottleneck convergence. *Mathematics of Operations Research* **38**, 469–491 (2013).

[2] G. Carinci, A. De Masi, C. Giardinà, E. Presutti, Hydrodinamic limit in a particle system with topological interactions. To appear in *Arabian Journal of Mathematics (2014)*, Preprint http://arxiv.org/abs/1307.6385.

[3] G. Carinci, A. De Masi, C. Giardinà, E. Presutti, Super-hydrodinamic limit in interacting particle system. Preprint http://arxiv.org/abs/1312.0640.

[4] L. Chayes, I. Kim, The supercooled Stefan problem in one dimension. *Communication on Pure and Applied Analysis* **11**, 845–859 (2012).

[5] L. Chayes, G. Swindle, Hydrodynamic limits for one-dimensional particle systems with moving boundaries. *Annals of Probability* **24** 559–598 (1996).

[6] F. Comets, A. De Masi, E. Presutti, in preparation.

[7] J. Crank and R. S. Gupta, A method for solving moving boundary problems in heat-flow using cubic splines or polynomials. *J. Inst. Math. Appl.* **10** 296–304 (1972).

[8] E. De Giorgi, New ideas in calculus of variations and geometric measure theory. Proceedings of the Conference *Motion by Mean Curvature and Related Topics*, held in Trento, 1992), pp. 63–69, Walter de Gruyter, Berlin (1994)

[9] A. De Masi, P.A. Ferrari, E. Presutti, Symmetric simple exclusion process with free boundaries. To appear in *Probability Theory and Related fields* (2014), Preprint http://arxiv.org/abs/1304.0701.

[10] A. De Masi, E. Presutti, D. Tsagkarogiannis, M.E. Vares, Current reservoirs in the simple exclusion process. *Journal of Statistical Physics* **144**, 1151–1170 (2011).

[11] A. De Masi, E. Presutti, D. Tsagkarogiannis, M.E. Vares, Truncated correlations in the stirring process with births and deaths. *Electronical Journal of Probability* **17**, 1–35, (2012).

[12] A. De Masi, E. Presutti, D. Tsagkarogiannis. Fourier law, phase transitions and the stationary Stefan problem. *Archive for Rational Mechanics and Analysis* **201**, 681–725 (2011).

[13] J. Douglas, A uniqueness theorem for the solution of a Stefan problem. *Proceedings of the American Mathematical Society* **8**(2), 402–408 (1957).

[14] A. Fasano, *Mathematical models of some diffusive processes with free boundaries.* SIMAI e-Lecture Notes (2008).

[15] A. Fasano and M. Primicerio, General free boundary problems for the heat equation, I. *J. Math. Anal. Appl.* **57** 694–723 (1977).

[16] A. Fasano and M. Primicerio, General free boundary problems for the heat equation, II. *J. Math. Anal. Appl.* **58** 202–231 (1977).

[17] A. Fasano and M. Primicerio, General free boundary problems for the heat equation, III. *J. Math. Anal. Appl.* **59** 1–14. (1977).

[18] A. Fasano and M. Primicerio, Free boundary problems for nonlinear parabolic equations with nonlinear free boundary conditions. *J. Math. Anal. Appl.* **72** 247–273 (1979).

[19] A. Fasano, M. Primicerio, S.D. Howison and J.R. Ockendon, Some remarks on the regularization of supercooled one-phase Stefan problems in one dimension. *Quart. Appl. Math.* **48** 153–168 (1990).

[20] A. Fasano, M. Primicerio, S.D. Howison and J.R. Ockendon, On the singularities of one-dimensional Stefan problems with supercooling. In Mathematical Models for Phase Change Problems, J.F. Rodrigues, editor. Int. Ser. Numerical Mathematics 88, 215–225 (1989), Birkhauser,

[21] A. Friedman, Free boundary problems for parabolic equations. Technical Report Report No. 28, Office of Naval Research (1958).

[22] A. Friedman, *Partial Differential Equations of Parabolic Type*, Holt, Reinhart, and Winston Inc., New York (1964).

[23] J.B.J. Fourier, *Théorie analytique de la chaleur*, Firmin-Didot, Paris (1822).

[24] H. Lacoin, The scaling limit of polymer pinning dynamics and a one dimensional Stefan freezing problem. Preprint arXiv:1204.1253 (2012).

[25] C. Landim and G. Valle, A microscopic model for Stefan melting and freezing problem. *Annals of Probability* **34**, 779–803 (2006).

[26] G.M. Lieberman, *Second order parabolic differential equations.* World scientific (1996).

[27] S. Luckhaus, Solutions for the two phase Stefan problem with the Gibbs Thompson law for the melting temperature. *European Journal of Applied Mathematics* **1**, 101–112 (1990).

[28] J. R. Ockendon, The role of the Crank-Gupta model in the theory of free and moving boundary problems. *Advances in Computational Mathematics* **6** 281–293 (1996)

[29] V.A. Roklin, On the fundamental ideas of measure theory. *Amer. Math. Soc. Translations* Series 1. **10**, 1–52 (1962).

[30] R. Srinivasan, Queues in Series via Interacting Particle. *Mathematics of Operations Research* **18**, 39–50 (1993).

[31] J. Stefan, Uber die Theorie der Eisbildung. *Monatshefte Mat. Phys.* **1**, 1–6 (1890).

[32] A. Visintin, Introduction to Stefan-type problems. In: *Handbook of Differential Equations: Evolutionary Differential Equations* vol. IV (C. Dafermos and M. Pokorny, eds.) North-Holland, Amsterdam, chap. 8, pp. 377–484 (2008).