Super-hydrodynamic limit in interacting particle systems

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Abstract

This paper is a follow-up of the work initiated in [3], where it has been investigated the hydrodynamic limit of symmetric independent random walkers with birth at the origin and death at the rightmost occupied site. Here we obtain two further results: first we characterize the stationary states on the hydrodynamic time scale and show that they are given by a family of linear macroscopic profiles whose parameters are determined by the current reservoirs and the system mass. Then we prove the existence of a super-hyrdrodynamic time scale, beyond the hydrodynamic one. On this larger time scale the system mass fluctuates and correspondingly the macroscopic profile of the system randomly moves within the family of linear profiles, with the randomness of a Brownian motion.

1 Introduction

In this paper we continue the analysis of the stochastic process introduced in [3]. This is a particles process in the interval $\Lambda_\epsilon := [0, \epsilon^{-1}] \cap \mathbb{Z}$, ϵ^{-1} a positive integer. The space of particles configurations is $\mathbb{N}^{\Lambda_\epsilon}$, $\xi = (\xi(x))_{x \in \Lambda_\epsilon} \in \mathbb{N}^{\Lambda_\epsilon}$ and the component $\xi(x) \in \mathbb{N}$ is interpreted as the number of particles at site x. The generator of the Markov process is

$$L = L^0 + L_b + L_a \tag{1.1}$$

(dependence on ϵ is not made explicit). L^0 is the generator of the independent random walks process with reflecting boundary conditions,

$$L^0 f(\xi) = \frac{1}{2} \sum_{x=0}^{\epsilon^{-1}-1} \xi(x) \left(f(\xi^{x,x+1}) - f(\xi) \right) + \xi(x+1) \left(f(\xi^{x+1,x}) - f(\xi) \right) \tag{1.2}$$

where $\xi^{x,y}$ denotes the configuration obtained from ξ by removing one particle from site x and putting it at site y. The operator L_b describes the action of creating a particle at the origin at rate ϵj, $j > 0$:

$$L_b f(\xi) = j\epsilon \left(f(\xi^+) - f(\xi) \right), \quad \xi^+(x) = \xi(x) + \mathbf{1}_{x=0} \ . \tag{1.3}$$

Instead L_a removes particles:

$$L_a f(\xi) = j\epsilon \left(f(\xi^-) - f(\xi) \right), \quad \xi^-(x) = \xi(x) - \mathbf{1}_{x=R_\xi} \tag{1.4}$$

namely a particle is taken out from the edge R_ξ of the configuration ξ:

$$R_\xi \text{ is such that: } \quad \begin{cases} \xi(y) > 0 & \text{for } y = R_\xi \\ \xi(y) = 0 & \text{for } y > R_\xi \ . \end{cases} \tag{1.5}$$

$L_a f(\xi) = 0$ if R_ξ does not exist, i.e. if $\xi \equiv 0$. The removal mechanism is therefore of topological nature, since the determination of the rightmost occupied site requires a knowledge of the entire configuration. Topological interactions appears in field as diverse as crowd dynamics [7] or swarm dynamics [1].

The independent random walkers process $\{\xi_t^0\}$, i.e. the process with generator L^0 and reflecting boundary conditions at 0 and ϵ^{-1}, can be thought as the evolution of an "isolated" system. The invariant measure for this process (when the total number n of particles is given) is a product of uniform distributions, i.e. each of the n particles occupy each of the $\epsilon^{-1} + 1$ sites with probability $1/(\epsilon^{-1} + 1)$. Moreover each particle equilibrates on times $\epsilon^{-2} t$ (convergence being exponentially fast in $\epsilon^{-2} t$).

The hydrodynamic limit for such an isolated system describes the behavior of the particles when $\epsilon \to 0$: the total number of particles is taken proportional to ϵ^{-1}, times are scaled by ϵ^{-2} while space is scaled down by ϵ (so that the macroscopic space is $[0,1] \subset \mathbb{R}$). It is well known [10] that the limit behavior (under suitable conditions on the initial configuration) is then given by the linear heat equation on $[0,1]$ with Neumann boundary conditions

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial r^2}, \qquad \frac{\partial \rho}{\partial r}\Big|_0 = \frac{\partial \rho}{\partial r}\Big|_1 = 0 \tag{1.6}$$

whose solution is $\rho_t(r) = G_t^{\text{neum}} * \rho_0(r) = \int G_t^{\text{neum}}(r,r')\rho_0(r')dr'$ where $G_t^{\text{neum}}(r,r')$, $r, r' \in [0,1]$, is the Green function of the heat equation (1.6) with Neumann boundary conditions:

$$G_t^{\text{neum}}(r,r') = \sum_k G_t(r, r'_k), \quad G_t(r,r') = \frac{e^{-(r-r')^2/2t}}{\sqrt{2\pi t}} \tag{1.7}$$

r'_k being the images of r' under repeated reflections of the interval $[0,1]$ to its right and left (see for instance [16] pag. 97 for details). The solution of (1.6) converges as $t \to \infty$ exponentially fast to the uniform distribution. Thus the hydrodynamic behavior given by (1.6) truly describes the behavior of the particles not only on times of order ϵ^{-2} (on which (1.6) is derived) but at all times as well: there is only one time scale in the isolated system. We will see that this is in contrast with the two time scales in the "open" system that we study here, where "open" means that the system is in contact with "the outside", i.e. particles can be created and killed.

The type of open systems most studied in the literature is that with "density reservoirs" [8] which impose an average density ρ_+ and ρ_- at the boundary sites (respectively 0 and ϵ^{-1}) via creation and annihilation of particles at both sides. By suitably defining such birth-death processes, a system of independent walkers reaches a stationary measure which is a product of Poisson distributions with average density which interpolates linearly the boundary densities $\rho_\pm$, see [5] for the finite size correction and also [9] where the result is proved for a class of zero range processes. In this case the hydrodynamic equation reads

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \qquad \rho(0) = \rho_+, \;\; \rho(1) = \rho_- \tag{1.8}$$

and the stationary profile is given by the linear profile in $[0,1]$ which interpolates between $\rho_\pm$. Again, also in the $t \to \infty$ limit, there is complete agreement between the hydrodynamic equations and the particles process. The system has still only one time scale.

The density reservoirs creates a non-equilibrium state with a current flowing through the system. By the continuity equation such macroscopic current is given by $-\frac{1}{2}\frac{\partial \rho}{\partial r}$ and in the stationary state of equation (1.8) one recovers Fick's law

$$-\frac{1}{2}\frac{\partial \rho}{\partial r} = \frac{\rho_+ - \rho_-}{2}\ .$$

At the microscopic level, the current generated by the density reservoirs is the difference between the average number of particles crossing a bond $(x, x+1)$ from the left and the average number of particles crossing it from the right. Thus it is equal to

$$\frac{E[\xi(x)] - E[\xi(x+1)]}{2} \approx \epsilon\, \frac{\rho_+ - \rho_-}{2}$$

(denoting here by E expectation with respect to the stationary measure and recalling that a particle jumps from x to $x+1$ and viceversa at rate $1/2$). Thus the micro-current is proportional to ϵ.

Another option to create a non-equilibrium state in an open system is to consider "current reservoirs" (see also [11, 12, 13, 14]). They are constructed in such a way to get directly a current ϵj just by throwing in particles from the left at rate ϵj and removing them from the right at same rate, without fixing the densities at the boundaries. This is obtained by the action of L_b in (1.3) and L_a in (1.4), which is to add from the left and respectively remove from the right particles at rate ϵj. As a result, the "current reservoirs" directly impose a current ϵj.

To better appreciate the role of current reservoirs in a non-equilibrium context it is useful to draw a parallelism with the problem of fixing a macroscopic quantity in equilibrium, for instance the magnetization in the Ising model. In that case one has two possibilities: either one introduces an external magnetic field which select a macroscopic state with the desired magnetization or one can choose from the very beginning to restrict the statistical average to the microscopic configurations compatible with the desired magnetization (micro-canonical ensemble). In a similar manner, to impose a given current in non-equilibrium system satisfying Fourier law, we can either fix the densities at the boundary (using density reservoirs) or, alternatively, restrict the system evolution to those trajectory with a prescribed current. This is precisely what the current reservoirs do.

In [3] the hydrodynamic limit of a system of symmetric independent walkers with current reservoirs, namely the process with generator (1.1), has been studied. The result established in that paper is the existence and continuity of the macroscopic profile when the microscopic process is started from a sufficiently nice initial configuration. The hydrodynamic scaling limit is characterized as the separating elements of upper and lower barriers (we give in section 2 a brief account of the results in [3]).

In the present paper we further investigate the macroscopic properties of the system. As a first result we compute the stationary macroscopic profiles in the hydrodynamic limit. We prove they are given by linear functions with slope $-2j$. Since here the boundary densities are not fixed we are in a situation with infinitely many such profiles. The one that is selected by the system is dictated by the total mass, which is a conserved quantity on the time scale $\epsilon^{-2}t$. However, on a longer time scale over which fluctuations of the total mass are allowed, there is not anymore a privileged profile and indeed the system will explore different profiles. Fluctuations of the total mass will occur on a *super-hydrodynamic* time scale. More precisely the super-hydrodynamic scaling is obtained by taking $\epsilon \to 0$ when the initial number of particles is taken proportional to ϵ^{-1}, times are scaled by ϵ^{-3} while space is scaled down by ϵ. We prove that in this limit the macroscopic profiles of the system moves randomly over the linear profiles with slope $-2j$ and the motion is controlled by the rescaled total mass which performs a Brownian motion reflected at the origin.

While in this paper we deal with independent random walkers we conjecture the phenomenon of the existence of a super-hydrodynamic scale in interacting particle systems coupled to current reservoirs to be quite universal. More precisely we claim the same phenomenon is to be expected for all systems (exclusion walkers, zero-range process, inclusion walkers, ...) which in the hydrodynamic limit scale to the free boundary problem given by

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} + jD_0 - jD_{R(t)} \tag{1.9}$$

where $R(t)$ is the macroscopic counterpart of the edge introduced in (1.5), D_0 denotes a Dirac delta at the origin corresponding to creation of particles, $D_{R(t)}$ denotes a Dirac delta at $R(t)$ corresponding to removal of the rightmost particles. This free boundary problem will be studied in [4] (see also [6, 17]). The two-time scales observed in our system is reminiscent of what is found in the context of processes with a localized schock, see for instance [2]. The peculiar and maybe surprising aspect of the super-hydrodynamic limit is the fact that on the time scale $\epsilon^{-3}t$ the system show persistent randomness, while on the hydrodynamic scale $\epsilon^{-2}t$ the system follows a deterministic evolution.

The paper is organized as follows. In section 2, after recalling the concept of partial order in the sense of mass transport and the construction of barriers introduced in [3], we state our main results: Theorem 2.3 which states that the hydrodynamic stationary profiles are the linear ones; Theorem 2.4 describing the profiles that in the course of time are attracted to the linear ones; Theorem 2.5 dealing with the super-hydrodynamic limit. In Section 3 we prove Theorem 2.3: we need to perform a separate analysis for the case with a non-trivial edge ($R(\infty) := R < 1$) and the case where the support of the stationary linear profile coincides with $[0, 1]$ ($R = 1$). In Section 4 we prove the remaining results. The convergence to linear profiles (Theorem 2.4) is obtained by introducing a coupling between two processes and showing that the number of discrepancies vanishes on the hydrodynamic scale; the evolution of profiles on the hydrodynamic time scale is proved by exploiting the convergence of the law of the mass density to the law of a Brownian motion on $\mathbb{R}^+$ reflected at the origin.

2 Definitions and main results

We consider initial configurations that approximate a macroscopic profile in the following sense. We first define the local empirical averages of a configuration $\xi \in \mathbb{N}^{\Lambda_\epsilon}$ and of a profile $\rho \in L^\infty([0,1],\mathbb{R}_+)$ as follows. Given any integer ℓ and $x \in [0, \epsilon^{-1} - \ell + 1]$, the empirical averages are

$$\mathcal{A}_\ell(x,\xi) := \frac{1}{\ell} \sum_{y=x}^{x+\ell-1} \xi(y) \qquad \text{and} \qquad \mathcal{A}'_\ell(x,\rho) = \frac{1}{\varepsilon\ell} \int_{\epsilon x}^{\epsilon(x+\ell)} \rho(r) dr \tag{2.1}$$

Definition 2.1. (Assumptions on the initial conditions.) We suppose $\rho_{\text{init}} \in C([0,1],\mathbb{R}_+)$ and, if it exists, we call $R(0) = \min\{r : \rho_{\text{init}}(r') = 0\ \forall r' \in [r,1]\}$, the "edge" of ρ_{init}. We fix $b < 1$ suitably close to 1 and $a > 0$ suitably small, we then denote by ℓ the integer part of ϵ^{-b} and suppose that for any $\epsilon > 0$ the initial configuration ξ verifies

$$\max_{x \in [0, \epsilon^{-1} - \ell + 1]} \Big| \mathcal{A}_\ell(x,\xi) - \mathcal{A}'_\ell(x, \rho_{\text{init}}) \Big| \le \epsilon^a . \tag{2.2}$$

We suppose moreover that, if ρ_{init} has an edge $R(0)$, then

$$|\epsilon R_\xi - R(0)| \le \epsilon^a \tag{2.3}$$

with R_ξ defined in (1.5). We shall denote by $P^{(\epsilon)}_\xi$ the law of the process with generator L given in (1.1) supported at time 0 by a configuration ξ as above.

Hydrodynamic limit.

The following Theorem has been proved in [3].

Theorem 2.1 (Existence of hydrodynamic limit)**.** *Let ρ_{init} and ξ be as in Definition 2.1. Then there exists a function $\rho_t(r) \ge 0$, $t \ge 0$, $r \in [0,1]$, equal to ρ_{init} at time $t = 0$, continuous in (r,t) and such that for all $T > 0$, $\zeta > 0$ and $t \in [0,T]$ the following holds*

$$\lim_{\epsilon \to 0} P^{(\epsilon)}_\xi \Big[\max_{x \in [0,\epsilon^{-1}]} |\epsilon F_\epsilon(x; \xi_{\epsilon^{-2}t}) - F(\epsilon x; \rho_t)| \le \zeta \Big] = 1 \tag{2.4}$$

where

$$F(r;u) = \int_r^1 u(r')\, dr', \qquad F_\epsilon(x;\xi) := \sum_{y \ge x} \xi(y) . \tag{2.5}$$

In particular for all smooth ϕ and for all $\zeta > 0$ one has

$$\lim_{\epsilon \to 0} P^{(\epsilon)}_\xi \left[\Big| \epsilon \sum_x \xi_{\epsilon^{-2}t}(x)\phi(x) - \int_0^1 \phi(r)\rho_t(r) dr \Big| \le \zeta \right] = 1 .$$

In [3] we have also proved that the limit profile ρ_t can be identified as the separating element between barriers, with the barriers defined as solutions of discrete Stefan problems. To explain this result, calling D_0 the Dirac delta at 0, we preliminary define the sets

$$\begin{aligned} \mathcal{U} &:= \Big\{ u = cD_0 + \rho : c \ge 0,\ \ \rho \in L^\infty([0,1],\mathbb{R}_+) \Big\} \\ \mathcal{U}_\delta &:= \Big\{ u = cD_0 + \rho : \int \rho > j\delta,\ c \ge 0,\ \ \rho \in L^\infty([0,1],\mathbb{R}_+) \Big\} \end{aligned} \tag{2.6}$$

and the *cut-and-paste* operator $K^{(\delta)} : \mathcal{U}_\delta \to \mathcal{U}$

$$K^{(\delta)}u = j\delta D_0 + \mathbf{1}_{r\in[0,R_u^{(\delta)}]}u, \qquad R_u^{(\delta)} : \int_{R_u^{(\delta)}}^1 u(r)dr = j\delta \ . \tag{2.7}$$

Definition 2.2 (Barriers)**.** Given $u \in L^\infty([0,1],\mathbb{R}_+)$ with $\int u > 0$ we define, for all δ small enough so that $u \in \mathcal{U}_\delta$, the "barriers" $S_{n\delta}^{(\delta,\pm)}(u)$, $n \in \mathbb{N}$, as follows: we set $S_0^{(\delta,\pm)}(u) = u$, and, for $n \geq 1$,

$$\begin{aligned} S_{n\delta}^{(\delta,-)}(u) &= K^{(\delta)}G_\delta^{\text{neum}} * S_{(n-1)\delta}^{(\delta,-)}(u) \\ S_{n\delta}^{(\delta,+)}(u) &= G_\delta^{\text{neum}} * K^{(\delta)}S_{(n-1)\delta}^{(\delta,+)}(u) \end{aligned} \tag{2.8}$$

where $G_t^{\text{neum}}(r,r')$, $r,r' \in [0,1]$, $t \geq 0$ is the Green function of the linear heat equation on $[0,1]$ with Neumann boundary conditions.

The functions $S_{n\delta}^{(\delta,\pm)}$ are obtained by alternating the map G_δ^{neum} (i.e. the heat kernel) and the cut and paste map $K^{(\delta)}$ (which takes out a mass $j\delta$ from the right and put it back at the origin, the macroscopic counterpart of L_b+L_a). It can be easily seen that, unlike the true process $(\xi_t)_{t\geq 0}$, the evolutions $S_{n\delta}^{(\delta,\pm)}$ conserve the total mass, that $S_{n\delta}^{(\delta,+)}$ maps $\mathcal{U}_\delta$ into L^∞ while $S_{n\delta}^{(\delta,-)}$ has a singular component $(j\delta D_0)$ plus a L^∞ component.

The evolutions $S_{n\delta}^{(\delta,\pm)}$ define barriers in the sense of the following partial order.

Definition 2.3. (Partial order). For u and v in the set $\mathcal{U}$ we define

$$u \leq v \qquad \text{iff} \qquad F(r;u) \leq F(r;v) \qquad \text{for all } r \in [0,1] \ . \tag{2.9}$$

where $F(r;\cdot)$ is defined in (2.5).

In [3] we have proved the following Theorem.

Theorem 2.2 (Hydrodynamic limit via barriers)**.** *Let ρ_t be the function of Theorem 2.1. Then ρ_t is the unique separating element between the barriers $\{S_{n\delta}^{(\delta,-)}(\rho_{\text{init}})\}$ and $\{S_{n\delta}^{(\delta,+)}(\rho_{\text{init}})\}$, namely for any $t > 0$, any $r \in [0,1]$ and any $n \in \mathbb{N}$:*

$$S_t^{(t2^{-n},-)}(\rho_{\text{init}}) \leq \rho_t \leq S_t^{(t2^{-n},+)}(\rho_{\text{init}}) \tag{2.10}$$

in the sense of (2.9). Furthermore the lower bound is a non decreasing function of n, the upper bound a non increasing function of n and

$$\lim_{n\to\infty} \sup_{r\in[0,1]} \left| F(r; S_t^{(t2^{-n},\pm)}(\rho_{\text{init}})) - F(r;\rho_t) \right| = 0 \tag{2.11}$$

Stationary profiles in the hydrodynamic time scale.

Our first result will be a full characterization of the stationary macroscopic states in the hydrodynamic limit.

Definition 2.4 (Linear profiles)**.** We denote by $\mathcal{M}$ "the manifold" of density profiles whose elements are either of the form (i) $\rho(r) = -2j(r-R)\mathbf{1}_{r\le R}$, $R \in (0,1)$; or (ii) $\rho(r) = -2jr + c$, $c \ge 2j$. They are conveniently parameterized as $\rho^{(M)}$, $M \ge 0$, where M is defined so that:

$$\int_0^1 \rho^{(M)}(r)dr = M, \qquad \rho^{(0)} \equiv 0 \tag{2.12}$$

In particular case (i) corresponds to $M < j$ and case (ii) is found for $M > j$.

Theorem 2.3 (Stationary profiles)**.** *If $\rho_{\text{init}} \in \mathcal{M}$ then $\rho_t = \rho_{\text{init}}$ for all $t \ge 0$.*

Super-hydrodynamic limit.

Hydrodynamics describes the behavior of the system on times $\epsilon^{-2}t$ in the limit when $\epsilon \to 0$. In our case the limit evolution is given by ρ_t as obtained in Theorem 2.1. Hydrodynamics predicts convergence to equilibrium:

Theorem 2.4 (Convergence to the stationary profiles)**.** *If $\int_0^1 \rho_{\text{init}}(r)dr = M$ then $\rho_t \to \rho^{(M)}$ in the sense that*

$$\lim_{t\to\infty} \sup_{r\in[0,1]} \left| F(r;\rho_t) - F(r;\rho^{(M)}) \right| = 0 \tag{2.13}$$

As a consequence of (2.13) and if ξ and ρ_{init} are as in Definition 2.1 then for any $\zeta > 0$

$$\lim_{t\to\infty} \lim_{\epsilon\to 0} P_\xi^{(\epsilon)} \Big[\max_{x\in[0,\epsilon^{-1}]} |\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t}) - F(\epsilon x;\rho^{(M)})| \ge \zeta \Big] = 0 \tag{2.14}$$

where $M = F(0;\rho_{\text{init}})$. (2.14) shows convergence in the hydrodynamic time scale to the invariant profiles of the limit evolution. There is here however an obvious interchange of limits as the convergence to the invariant profile is only after taking the limit $\epsilon \to 0$. The true long time particle behavior requires instead the study of the process $\xi_{\epsilon^{-2}t_\epsilon}$ where $t_\epsilon \to \infty$ as $\epsilon \to 0$. If in this limit we obtain something different than (2.14) then we say that there are other scales than the hydrodynamical one, that we call super-hydrodynamic.

Theorem 2.5 (Super-hydrodynamic limit)**.** *Let $\xi^{(\epsilon)}$ be a sequence such that $\epsilon|\xi^{(\epsilon)}| \to m > 0$ as $\epsilon \to 0$. Let t_ϵ be an increasing, divergent sequence, then the process $\xi_{\epsilon^{-2}t_\epsilon}$ has two regimes:*

- *Subcritical. Suppose $\epsilon t_\epsilon \to 0$, then*

$$\lim_{\epsilon\to 0} P_{\xi^{(\epsilon)}}^{(\epsilon)} \Big[\max_{x\in[0,\epsilon^{-1}]} |\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t_\epsilon}) - F(\epsilon x;\rho^{(m)})| \le \zeta \Big] = 1 \tag{2.15}$$

- *Critical. Let $t_\epsilon = t\epsilon^{-1}$ then*

$$\lim_{\epsilon\to 0} P_{\xi^{(\epsilon)}}^{(\epsilon)} \left[\max_{x\in[0,\epsilon^{-1}]} |\epsilon F_\epsilon(x;\xi_{\epsilon^{-3}t}) - F(\epsilon x;\rho^{(M_t^{(\epsilon)})})| \le \zeta \right] = 1 \tag{2.16}$$

where $M_t^{(\epsilon)}$ converges in law as $\epsilon \to 0$ to B_{jt}, where $(B_t)_{t\ge 0}$, $B_0 = m$, is the Brownian motion on $\mathbb{R}_+$ with reflections at the origin.

Thus on a first time scale, i.e. the subcritical regime, the process behaves deterministically, it is attracted by the manifold $\mathcal{M}$ and equilibrates to one of the invariant profiles for the limit evolution, the one with the same mass. However on longer times of the order $\epsilon^{-3}t$ it starts moving stochastically on the manifold $\mathcal{M}$ where it performs a Brownian motion. The reason is pretty simple because the total number $|\xi_t|$ of particles at time t performs a symmetric random walk reflected at the origin:

Theorem 2.6 (Distribution of the particles' number). $|\xi_t| = \sum_{x=0}^{\epsilon^{-1}} \xi_t(x)$ *has the law of a continuous time random walk on* $\mathbb{N}$ *which jumps with equal probability by* ± 1 *after an exponential time of parameter* $2j\epsilon$*, the jumps leading to* -1 *being suppressed.*

3 Stationary macroscopic profiles

In this section we shall study the fixed point of $S_\delta^{(\delta,-)}$ (see Definition 2.2) and their limits as $\delta \to 0$. We will show that the stationary profiles are linear in this limit.

3.1 The case $R < 1$

We first analyze the case when the total mass in less than j that yields profiles with support in $[0, R]$ with $R < 1$.

Theorem 3.1. *For any* $R \in (0,1)$ *and any* $\delta > 0$ *small enough there is a unique, continuous function* $\rho \geq 0$*, hereafter called "stationary profile", with support in* $[0, R]$*,* $R < 1$*, and such that*

$$S_\delta^{(\delta,-)}(j\delta D_0 + \rho) = j\delta D_0 + \rho \tag{3.1}$$

Moreover ρ *is an increasing function of* R*.*

Proof. By (2.8)

$$S_\delta^{(\delta,-)}(u) = j\delta D_0 + G_\delta^{\text{neum}} * u \cdot \mathbf{1}_{r\in[0,x]}, \quad x = R^{(\delta)}_{G_\delta^{\text{neum}}*u}$$

If u is a fixed point of $S_\delta^{(\delta,-)}$, i.e. $S_\delta^{(\delta,-)}(u) = u$, then $u = j\delta D_0 + \rho$ with the support of $\rho = G_\delta^{\text{neum}} * u$ being the interval $[0, x]$. As we look for solutions with support in $[0, R]$ we must take $x = R$ and thus get for ρ the equation

$$\rho(r) = j\delta G_\delta^{\text{neum}}(0, r) + \int_0^R dr' G_\delta^{\text{neum}}(r', r)\rho(r'), \quad r \in [0, R] \tag{3.2}$$

The last condition in (2.7) (with $x \to R$) becomes:

$$\int_R^1 dr' \int_0^R dr G_\delta^{\text{neum}}(r', r)[\rho(r) + j\delta D_0(r)] = j\delta \tag{3.3}$$

However (3.3) is not an extra condition as it is automatically satisfied if ρ satisfies (3.2):

$$\begin{aligned}
&\int_R^1 dr' \int_0^R dr G_\delta^{\mathrm{neum}}(r',r)[\rho(r)+j\delta D_0(r)] \\
&= \int_0^1 dr' \int_0^R dr G_\delta^{\mathrm{neum}}(r',r)[\rho(r)+j\delta D_0(r)] - \int_0^R dr' \int_0^R dr G_\delta^{\mathrm{neum}}(r',r)[\rho(r)+j\delta D_0(r)] \\
&= j\delta + \int_0^R \rho - \int_0^R \rho
\end{aligned}$$

The proof of the theorem is then a consequence of the following lemma.

Lemma 3.2. *Call*

$$g_\delta^0(r,r') = \mathbf{1}_{r,r'\in[0,R]} G_\delta^{\mathrm{neum}}(r,r'), \quad g_{n\delta}^0 = g_\delta^0 * \cdots * g_\delta^0, \quad (n \text{ times}) \tag{3.4}$$

Then the series

$$j\delta \sum_{n\ge 0} g_{(n+1)\delta}^0(0,r) =: \rho(r), \quad r\in[0,R] \tag{3.5}$$

is uniformly convergent in r and δ, so that ρ (defined by (3.5)*) is the unique solution of* (3.2)*.*

Proof. To prove convergence we observe that there is a positive constant a such that

$$\sup_{r\in[0,R]} \int g_{\delta N_\delta}^0(r,r')dr' \le 1-a, \quad N_\delta\in\mathbb{N}: \;\; \delta(N_\delta-1) < 1 \le \delta N_\delta \tag{3.6}$$

(a can be taken as the sup of the probability that a Brownian motion on $\mathbb{R}$ which starts at $r\in[0,R]$ is in $(R,1)$ at time δN_δ). We have

$$g_{n\delta}^0(r,r') \le \frac{c}{\sqrt{n\delta}}, \quad \text{for all } n$$

Then

$$j\delta \sum_{n=0}^{N_\delta} g_{(n+1)\delta}^0(0,r) \le c'$$

It follows from (3.6) that

$$g_{n\delta}^0(r,r') \le (1-a)^{k-1} \sup_{r''} g_{\delta(m+N_\delta)}^0(r'',r'), \quad n = kN_\delta + m,\; k\ge 1,\; 0\le m < N_\delta \tag{3.7}$$

with $g_{\delta(m+N_\delta)}^0(r'',r') \le c''$. Thus

$$j\delta \sum_{n>N_\delta} g_{(n+1)\delta}^0(0,r) \le j\delta \sum_{k\ge 1}\sum_{m<N_\delta} c''(1-a)^{k-1} \le c'''$$

this proves the Lemma. □

Continuity of ρ follows from (3.5). To prove that ρ increases with R we use the following representation of the Green function g^0: let $I=[a,a']$ be an interval in $[0,R]$, $J=[R,1]$, I^* and J^* the union of the repeated reflections of I and J around 0 and 1. Then

$$\int_I g_{n\delta}^0(r,r')dr' = P_r\Big[B_{n\delta}\in I^*, \;\; B_{k\delta}\notin J^*, k\le n\Big] \tag{3.8}$$

where P_r is the law of the Brownian motion $(B_s)_{s\ge 0}$ on $\mathbb{R}$ which starts from $r\in[0,R]$. The right hand side is clearly increasing with R. This concludes the proof of Theorem 3.1. □

Theorem 3.3. *Let $\rho^{(\delta,-)} := j\delta D_0 + \rho$, ρ as in Theorem 3.1, then*

$$\lim_{\delta\to 0} \rho^{(\delta,-)}(r) = 2j(R-r), \quad r \in [0,R] \tag{3.9}$$

Proof. The proof is in two steps: in the first one we prove that the series in (3.5) converges as $\delta \to 0$ (it is, approximately, a Riemann sum of an integral) while in the second step we recognize the limit to be the linear function in (3.9). We proceed by proving lower and upper bounds which in the limit $\delta \to 0$ will coincide.

Lower bound. Let $I = [a,a'] \subset [0,R]$, then by (3.8)

$$\int_a^{a'} g_t^0(r,r')dr' \ge P_r\Big[B_t \in \{[a,a'] \cup [-a',-a]\},\ \sup_{s\le t}|B_s| \le R\Big], \quad t = n\delta \tag{3.10}$$

Thus denoting by $G_t^{\mathrm{Dir}}(r,r')$ the Green function of the heat equation $u_t = \frac{1}{2}u_{rr}$ in $[-R,R]$ with Dirichlet boundary conditions $u(\pm R) = 0$:

$$\rho^{(\delta,-)}(r) \ge j\delta \sum_{n\ge 0}\Big(G^{\mathrm{Dir}}_{(n+1)\delta}(0,r) + G^{\mathrm{Dir}}_{(n+1)\delta}(0,-r)\Big) \tag{3.11}$$

The right hand side is the Riemann sum of the corresponding integral and due to the uniform convergence of the series proved earlier we have

$$\liminf_{\delta\to 0} \rho^{(\delta,-)}(r) \ge j\int_0^\infty \Big(G_t^{\mathrm{Dir}}(0,r) + G_t^{\mathrm{Dir}}(0,-r)\Big)\,dt \tag{3.12}$$

Let $v(s,r)$ be the resolvent of the heat equation with Dirichlet boundary conditions in $[-R,R]$, then v verifies the resolvent equation $\frac{1}{2}v_{rr} + D_0 = sv$. Hence the integral $\displaystyle\int_0^\infty G_t^{\mathrm{Dir}}(0,r) =$ $v(0,r) := v_0(r)$ is the weak solution of the problem $\dfrac{1}{2}v_{rr} + D_0 = 0$, $v(\pm R) = 0$, namely $v_0(r) = R - |r|$, $r \in [-R,R]$. Then, from (3.12),

$$\liminf_{\delta\to 0} \rho^{(\delta,-)}(r) \ge 2jv_0(r) = 2j(R-|r|) \tag{3.13}$$

this proves that the lower bound agrees with (3.9).

Upper bound. We first observe that there are positive constants α and β so that for all δ small enough:

$$P_0\Big[\sup_{s\le\delta}|B_s| \ge \delta^{1/2-\alpha}\Big] \le e^{-\beta\delta^{-2\alpha}} \tag{3.14}$$

and get from (3.8) with $R_\delta := R + \delta^{1/2-\alpha}$

$$\int_a^{a'} g^0_{n\delta}(r,r')dr' \le P_r\Big[B_{n\delta} \in \{[a,a'] \cup [-a',-a]\}),\ \ |B_s| \le R_\delta, s \le n\delta\Big] + ne^{-\beta\delta^{-2\alpha}} \tag{3.15}$$

We use (3.15) for $n \le \delta^{-2}$ and get, recalling (3.7),

$$\rho^{(\delta,-)}(r) \le j\delta\Big(\sum_{n=0}^{\delta^{-2}}[G^{\mathrm{Dir},R_\delta}_{(n+1)\delta}(0,r) + G^{\mathrm{Dir},R_\delta}_{(n+1)\delta}(0,-r)] + 2\delta^{-4}e^{-\beta\delta^{-2\alpha}}\Big) + c(1-a)^{\delta^{-1}} + j\delta D_0 \tag{3.16}$$

where $G^{\mathrm{Dir},R_\delta}$ is the Green function with Dirichlet boundary conditions in $[-R_\delta, R_\delta]$ and c is a suitable constant. By letting $\delta \to 0$ we recover the lower bound, we omit the details. $\square$

3.2 The case $R = 1$

The analysis so far covers cases where the limit profile is a piecewise linear function with slope $-2j$ in $[0, R]$, $R < 1$, and equal to 0 in $[R, 1]$. The mass is therefore jR^2, hence the analysis does not apply to cases where the mass is $> j$. As we shall see a posteriori this corresponds to stationary solutions for the $(\delta, -)$ evolution having support of the form $[0, R]$, with $R = 1 - A\delta$. We are going to prove that the analogue of Theorem 3.1 holds as well when $R = 1 - A\delta$, $A > 0$ and δ small enough.

Equations (3.2)–(3.5) hold unchanged but (3.6) needs a new proof. Calling P_r the law of the Brownian motion B_t on $\mathbb{R}$ which starts from $r \in [0, R]$, we have:

$$\int g^0_{\delta N_\delta}(r, r')dr' = P_r\Big[B_{n\delta} \notin J^*, \;\; n = 1, .., N_\delta\Big] \tag{3.17}$$

where $J = [1 - A\delta, 1]$ and J^* is the union of all reflections of J.

Lemma 3.4. *There is $a > 0$ so that for any δ small enough*

$$\int g^0_{\delta N_\delta}(r, r')dr' \leq 1 - a, \quad \textit{for all } r \in [0, 1 - A\delta] \tag{3.18}$$

Proof. By (3.17)

$$\begin{aligned}
\int g^0_{\delta N_\delta}(r, r')dr' &= P_r\Big[B_{n\delta} \notin J^*, \;\; n = 1, .., N_\delta\Big] \leq P_r\Big[B_{n\delta} \notin J, \;\; n = 1, .., N_\delta\Big] \\
&= P_r[X = 0], \quad X := \sum_{n=1}^{N_\delta} \mathbf{1}_{B_{n\delta} \in J}
\end{aligned} \tag{3.19}$$

Let

$$p_k = P_r\big[X = k\big], \quad M_i = \sum_{k \geq 1} p_k k^i, \;\; i = 0, 1, 2 \tag{3.20}$$

so that $P_r[X = 0] = 1 - M_0$. Hence, by (3.19), we can take for a in (3.18) any lower bound for M_0. We are going to show that

$$M_0 \geq \frac{M_1^2}{2(2M_2 + M_1)} \geq \frac{M_1^2}{6M_2} \tag{3.21}$$

We have

$$M_2 \geq \sum_{k \geq k_0} p_k k^2 \geq k_0 \sum_{k \geq k_0} p_k k$$

We choose k_0 to be the smallest integer so that

$$\frac{M_2}{k_0} \leq \frac{M_1}{2}, \quad \frac{2M_2}{M_1} \leq k_0 \leq \frac{2M_2}{M_1} + 1$$

Then

$$k_0 \sum_{k \leq k_0} p_k \geq \sum_{k \leq k_0} k p_k = M_1 - \sum_{k > k_0} k p_k \geq M_1 - \frac{M_2}{k_0} \geq \frac{M_1}{2}$$

Thus

$$M_0 \geq \sum_{k \leq k_0} p_k \geq \frac{M_1}{2k_0} \geq \frac{M_1^2}{2(2M_2 + M_1)}$$

(3.21) is thus proved.

We have

$$M_1 = \sum_{n=1}^{N_\delta} \int_J \frac{e^{-(r-r')^2/(2\delta n)}}{\sqrt{2\pi\delta n}} dr'$$

thus

$$M_1 \geq \frac{e^{-c}\delta}{\sqrt{2\pi\delta}} \sum_{n=N_\delta/2}^{N_\delta} n^{-1/2} \geq \frac{e^{-c}\delta}{\sqrt{2\pi\delta}} \frac{\sqrt{N_\delta}}{2} \geq C_1$$

where c and C_1 are constant independent of r and δ (recall that $N \approx \delta^{-1}$). An analogous proof yields $M_1 \leq C_2$, C_2 a constant independent of r and δ. Moreover

$$M_2 = M_1 + \sum_{1 \leq n_1 < n_2 \leq N_\delta} \int_J dr' \int_J dr'' \ \frac{e^{-(r-r')^2/(2\delta n_1)}}{\sqrt{2\pi\delta n_1}} \ \frac{e^{-(r'-r'')^2/(2\delta n_2)}}{\sqrt{2\pi\delta n_2}}$$

As before we can prove (details are omitted) that $M_2 \leq C_3$, a constant independent of r and δ.

Since $P_r[X = 0] = 1 - M_0$ the above together with (3.19) proves the lemma with

$$a = \frac{C_1^2}{6C_3}$$

□

After (3.6) the proof of Theorem 3.1 extends unchanged to the present case, so that the conclusions of Theorem 3.1 hold as well when $R = 1 - A\delta$. The analogue of Theorem 3.3 is:

Theorem 3.5. *Denoting by $\rho^{(\delta,-)}$ the "stationary profile" when $R = 1 - A\delta$, then for all $r \in [0, 1)$*

$$\lim_{\delta \to 0} \rho^{(\delta,-)}(r) = 2j(1 - r) + \rho(1), \quad \rho(1) := \frac{j}{A} \tag{3.22}$$

Proof. The main difference with Theorem 3.3 is that now we have to deal with an interval $[R, 1]$ which depends on δ and which shrinks to zero as $\delta \to 0$. We can however set the problem in such a way that the interval is the whole $[0, 1]$ for all δ. To this end we introduce another map $T_\delta^{(\delta,-)}$ which, for a special choice of the parameters, will have the same fixed points as $S_\delta^{(\delta,-)}$. Given a non negative function v we set

$$T_\delta^{(\delta,-)}(u) = j\delta D_0 - v + G_\delta^{\text{neum}} * u \tag{3.23}$$

Then a fixed point u must have the form: $u = j\delta D_0 - v + \psi(r)$ with ψ such that

$$\psi = G_\delta^{\text{neum}} * [j\delta D_0 - v + \psi] \tag{3.24}$$

where, as before, G_t^{neum} is the Green function of the linear heat equation in $[0, 1]$ with Neumann boundary conditions.

It is readily seen that if $\rho = \rho^{(\delta,-)} = \rho^{(\delta,-)}\mathbf{1}_{r\notin[1-A\delta,1]}$ and

$$v(r) := \mathbf{1}_{r\in[1-A\delta,1]} G_\delta^{\rm neum} * [j\delta D_0 + \rho](r) \tag{3.25}$$

then

$$\psi(r) := \begin{cases} \rho(r) & \text{if } r\in[0,1-A\delta] \\ v(r) & \text{if } r\in[1-A\delta,1] \end{cases} \tag{3.26}$$

solves (3.24).

On the other hand (3.24) can be solved by iteration getting, analogously to (3.5),

$$\psi(r) = \sum_{n\geq 0} \{ j\delta G^{\rm neum}_{(n+1)\delta}(r,0) - \int_{1-A\delta}^1 G^{\rm neum}_{(n+1)\delta}(r,r')v(r')\} \tag{3.27}$$

but again we need a proof that the series is convergent. The Green function converges exponentially:

$$|G_t^{\rm neum}(r,r') - 1| \leq c e^{-bt}, \quad c>0, \;\; b>0 \tag{3.28}$$

Moreover, by its definition, see (3.3),

$$\int_{1-A\delta}^1 v(r)dr = j\delta \tag{3.29}$$

Then

$$\Big| j\delta G^{\rm neum}_{n\delta}(r,0) - \int_{1-A\delta}^1 G^{\rm neum}_{n\delta}(r,r')v(r')\Big| \leq c' e^{-bn\delta}$$

so that the series (3.27) converges exponentially uniformly in δ.

Let us now add a superscript $(\delta,-)$ to ψ and v to underline their dependence on δ. We shall first prove that $\psi^{(\delta,-)}$ is equicontinuous:

Lemma 3.6. *For any $\epsilon>0$ there is $\alpha>0$ so that for all δ*

$$\sup_{|r-r'|\leq\alpha} |\psi^{(\delta,-)}(r) - \psi^{(\delta,-)}(r')| \leq \epsilon \tag{3.30}$$

Proof. By (3.28) given any $\epsilon>0$ there is $T>0$ so that

$$\sum_{n:n\delta\geq T} |j\delta G^{\rm neum}_{(n+1)\delta}(r,0) - \int_{1-A\delta}^1 G^{\rm neum}_{(n+1)\delta}(r,r'')v^{(\delta,-)}(r'')| \leq \epsilon \tag{3.31}$$

It is well known that for any $\zeta>0$ and $\tau>0$ there is $\alpha>0$ so that

$$\sup_{t\geq\tau}\;\sup_{|r-r'|\leq\alpha}\;\sup_{r''} |G_t^{\rm neum}(r,r'') - G_t^{\rm neum}(r',r'')| \leq \zeta$$

By bounding $G_t^{\rm neum}(r,r') \leq \dfrac{c}{\sqrt{t}}$

$$|\psi^{(\delta,-)}(r) - \psi^{(\delta,-)}(r')| \;\;\leq\;\; 2\epsilon + 4\sum_{n\delta\leq\tau} \frac{j\delta c}{\sqrt{n\delta}} + \delta^{-1}T2j\delta\zeta$$

By choosing $\zeta = \epsilon/T$ and $\tau = \epsilon^2$ we then have the right hand side bounded proportionally to ϵ and the lemma is proved. $\square$

By (3.31) and the lemma we have that for any ϵ and for all δ small enough:

$$\left|\psi^{(\delta,-)}(r) - j\delta \sum_n \left(G^{\text{neum}}_{(n+1)\delta}(r,0) - G^{\text{neum}}_{(n+1)\delta}(r,1)\right)\right| \le \epsilon \tag{3.32}$$

so that

$$\lim_{\delta\to 0} \psi^{(\delta,-)}(r) = \psi(r) = \int_0^\infty \{G^{\text{neum}}_t(r,0) - G^{\text{neum}}_t(r,1)\} \tag{3.33}$$

which proves that $\rho^{(\delta,-)}(r)$ converges to $\psi(r)$ for all $r < 1$. As in the previous case with $R < 1$ fixed, the right hand side is identified to be a weak solution of the equation

$$\psi'' + jD_0 - jD_1 = 0 \tag{3.34}$$

on $\mathbb{R}$ symmetric under all reflections of $[0,1]$. To determine the solution we need another condition, we are going to prove that at the right endpoint

$$A\psi(1) = j \tag{3.35}$$

Indeed,

$$j\delta = \int_{1-A\delta}^1 v^{(\delta,-)}(r)dr = A\delta v^{(\delta,-)}(1) + \int_{1-A\delta}^1 [v^{(\delta,-)}(r) - v^{(\delta,-)}(1)]dr$$

Recalling (3.26), $v^{(\delta,-)}(r) = \psi^{(\delta,-)}(r)$, $r \in (1-A\delta, 1)$, hence

$$|j\delta - A\delta\psi^{(\delta,-)}(1)| \le A\delta \sup_{1-A\delta\le r\le 1} |\psi^{(\delta,-)}(r) - \psi^{(\delta,-)}(1)|$$

By (3.30) in the limit as $\delta \to 0$ we then obtain (3.35). The weak solution of (3.34) with the condition (3.35) is the function on the right hand side of (3.22). $\square$

3.3 Stationarity of the linear profiles

Proof of Theorem 2.3. To underline the choice of the initial datum we denote the limit profile ρ_t of Theorem 2.1 by $\rho_t = S_t(\rho_{\text{init}})$. We fix $\tau > 0$ and have by Theorem 2.2

$$\lim_{n\to\infty} \left|F(r; S_\tau(\rho^{(M)})) - F(r; S_\tau^{(\tau 2^{-n},-)}(\rho^{(M)}))\right| = 0, \quad \text{for all } r \in [0,1] \tag{3.36}$$

Denote by $\rho^{(\tau 2^{-n},-)}$ the stationary profile for the evolution $S_t^{(\tau 2^{-n},-)}$ which converges to $\rho^{(M)}$, then

$$\left|F(r; S_\tau^{(\tau 2^{-n},-)}(\rho^{(\tau 2^{-n},-)})) - F(r; S_\tau^{(\tau 2^{-n},-)}(\rho^{(M)}))\right| \le \|S_\tau^{(\tau 2^{-n},-)}(\rho^{(\tau 2^{-n},-)})) - S_\tau^{(\tau 2^{-n},-)}(\rho^{(M)}))\|_1 \tag{3.37}$$

Since G^{neum}_t is a contraction in L_1 as well as $K^{(\delta)}$ (see Lemma (7.3) in [3]) we have

$$\|S_\tau^{(\tau 2^{-n},-)}(\rho^{(\tau 2^{-n},-)})) - S_\tau^{(\tau 2^{-n},-)}(\rho^{(M)}))\|_1 \le \|\rho^{(\tau 2^{-n},-)} - \rho^{(M)}\|_1 \tag{3.38}$$

which, from Theorems 3.3 and 3.5, vanishes as $n \to \infty$. Since $S_\tau^{(\tau 2^{-n},-)}(\rho^{(\tau 2^{-n},-)}) = \rho^{(\tau 2^{-n},-)}$,

$$\lim_{n\to\infty} \Big| F(r; S_\tau(\rho^{(M)})) - F(r; \rho^{(\tau 2^{-n},-)}) \Big| = 0, \quad \text{for all } r \in [0,1] \tag{3.39}$$

which concludes the proof because, as already observed,

$$\lim_{n\to\infty} \Big| F(r; \rho^{(M)}) - F(r; \rho^{(\tau 2^{-n},-)}) \Big| = 0, \quad \text{for all } r \in [0,1] \tag{3.40}$$

□

4 Super-hydrodynamic limit

The main result in this section is a proof of a loss of memory of the initial conditions on long hydrodynamic times. This result will be obtained by introducing couplings and to this end it will be convenient to label the particles. We shall then conclude the section by using the loss of memory result to prove convergence to linear stationary profiles and control the super-hydrodynamic limit.

Definition 4.1 (Labeled configurations)**.** A labeled configuration is a pair $(\underline{x}, I)$ where I is a finite subset of $\mathbb{N}$ and $\underline{x}$ a map from I to $[0, \epsilon^{-1}]$: I are the labels and $\underline{x}$ the positions of the labeled particles. We shall also write $\underline{x} = \{x_i,\, i \in I\}$. To any labeled configuration $(\underline{x}, I)$ we associate the unlabeled configuration $\xi_{\underline{x},I}$:

$$\xi_{\underline{x},I}(x) = \sum_{i\in I} \mathbf{1}_{x_i = x} \tag{4.1}$$

We shall couple the evolution starting from $(\underline{x}_0, I_0)$ and $(\underline{y}_0, J_0)$ where $I_0 = \{1, .., n\}$ and $J_0 = (1, ..., n+m)$, $n > 0$, $m \geq 0$. The coupled process will be a jump Markov process on a state space S which is the family of all $(\underline{x}, I, \underline{y}, J, N)$ such that $I \subset J$, $J \setminus I$ has cardinality $\leq m$ and $N = \max\{i \in J\}$.

The coupled process starts from $(\underline{x}_0, I_0, \underline{y}_0, J_0, n+m)$ and it is completely defined once we specify the possible jumps and their intensities starting from any element $(\underline{x}, I, \underline{y}, J, N)$ in the state space S. To this end we introduce the set

$$I_= = \{i \in I : x_i = y_i\}$$

and call $(\underline{x}', I', \underline{y}', J', N')$ the elements after the jump. The jumps are of four types:

- *Single random walk jumps.* They are independent random walk jumps involving the restricted configurations $(\underline{x}, I \setminus I_=)$ and $(\underline{y}, J \setminus I_=)$. For any of these jumps it will be $I' = I$, $J' = J$, $N' = N$. The jumps indexed by $i \in I \setminus I_=$ are such that $\underline{y}' = \underline{y}$ and $x'_j = x_j$ for $j \neq i$, while $x_i \to x_i \pm 1$ with intensity $1/2$ and $x'_i = x_i \pm 1$ if this is in $[0, \epsilon^{-1}]$, otherwise $x'_i = x_i$. Analogously the jumps indexed by $i \in J \setminus I_=$ are such that $\underline{x}' = \underline{x}$ and $y'_j = y_j$ for $j \neq i$, while $y_i \to y_i \pm 1$ with intensity $1/2$ and $y'_i = y_i \pm 1$ if this is in $[0, \epsilon^{-1}]$, otherwise $y'_i = y_i$. We denote by $\mathcal{L}_s$ the Markov generator describing the single random walk jumps. It is given by:

$$
\begin{aligned}
\mathcal{L}_s f(\underline{x}, \underline{y}) \quad = \quad & \sum_{i \in I \setminus I_=} \frac{1}{2} \left\{ \left(f(\underline{x}^{i,+}, \underline{y}) - f(\underline{x}, \underline{y}) \right) \mathbf{1}_{\{x_i \neq \epsilon^{-1}\}} + \left(f(\underline{x}^{i,-}, \underline{y}) - f(\underline{x}, \underline{y}) \right) \mathbf{1}_{\{x_i \neq 0\}} \right\} \\
+ \quad & \sum_{i \in J \setminus I_=} \frac{1}{2} \left\{ \left(f(\underline{x}, \underline{y}^{i,+}) - f(\underline{x}, \underline{y}) \right) \mathbf{1}_{\{y_i \neq \epsilon^{-1}\}} + \left(f(\underline{x}, \underline{y}^{i,-}) - f(\underline{x}, \underline{y}) \right) \mathbf{1}_{\{y_i \neq 0\}} \right\}
\end{aligned}
$$

where $\underline{x}^{i,\pm}$ is the positions configuration obtained from $\underline{x}$ by replacing x_i with $x_i \pm 1$. We have omitted to underline the dependence of f on I, J, N since they remain unchanged under the action of $\mathcal{L}_s$.

- *Double random walk jumps.* They are indexed by $i \in I_=$ and also for these jumps $I' = I$, $J' = J$, $N' = N$. For each $i \in I_=$, $x_i \to x'_i = x_i \pm 1$ and $y_i \to x'_i$, with intensity $1/2$ if $x_i \pm 1 \in [0, \epsilon^{-1}]$, otherwise the jump is suppressed; all the other positions are unchanged. Let $\mathcal{L}_d$ be the Markov generator describing the double random walk jumps, then it is given by:

$$
\begin{aligned}
&\mathcal{L}_d f(\underline{x}, \underline{y}) = \\
&\quad = \sum_{i \in I_=} \frac{1}{2} \left\{ \left(f(\underline{x}^{i,+}, \underline{y}^{i,+}) - f(\underline{x}, \underline{y}) \right) \mathbf{1}_{\{x_i, y_i \neq \epsilon^{-1}\}} + \left(f(\underline{x}^{i,-}, \underline{y}^{i,-}) - f(\underline{x}, \underline{y}) \right) \mathbf{1}_{\{x_i, y_i \neq 0\}} \right\}
\end{aligned}
$$

- *Creation events.* At rate ϵj, $N' = N+1$, $I' = I \cup \{N+1\}$, $J' = J \cup \{N+1\}$, $x'_i = x_i, i \in I$, $x'_{N+1} = 0$; $y'_i = y_i, i \in J$, $y'_{N+1} = 0$. We call $\mathcal{L}_{\mathrm{cr}}$ the Markov generator associated to these events.

- *Death events.* At rate ϵj both I and J loose an element while N is unchanged. The configuration after the death event is obtained in two steps. In the first step we erase from $\underline{x}$ and $\underline{y}$ their rightmost particle with largest label, say x_i and y_j. That is also the final step if $j \notin I$ or if $i = j$. If instead $i \neq j$ and $j \in I$ we have two subcases: if $x_j \le y_i$ we relabel y_i as y_j so that the label i disappears from I and J. If instead $y_i < x_j$ we relabel x_j as x_i so that the label j disappears from I and J. We denote by $\mathcal{L}_{\mathrm{ann}}$ the Markov generator associated to the death events.

It directly follows from the above rules that:

Lemma 4.1. *In all the above cases $(\underline{x}', I', \underline{y}', J', N') \in S$ and the set $I \setminus I_=$ does not increase after any of the above jumps. Moreover in the case of a death event, if $i \in I \cap I'$, the interval with endpoints x_i and y_i may only change in such a way that the distance $|x_i - y_i|$ decreases.*

One can then easily check that

Lemma 4.2. *The above rules can be used to define a jump process with state space S, denoted by $(\underline{x}(t), I(t), \underline{y}(t), J(t), N(t))$. Its generator $\mathcal{L}$ is*

$$
\mathcal{L} = \mathcal{L}_s + \mathcal{L}_d + \mathcal{L}_{\mathrm{cr}} + \mathcal{L}_{\mathrm{ann}} \tag{4.2}
$$

where $\mathcal{L}_s$ describes the single random walk jumps; $\mathcal{L}_d$ the double random walk jumps; $\mathcal{L}_{\mathrm{cr}}$ the creation and $\mathcal{L}_{\mathrm{ann}}$ the annihilation jumps.

The processes $\xi_{\underline{x}(t), I(t)}$ and $\xi_{\underline{y}(t), J(t)}$ are then both Markov with generator L defined in (1.1).

We say that i is a discrepancy at time t if it belongs to the set

$$D_{\neq}(t) = I(t) \setminus I_{=}(t) = \Big\{ i \in I(t) : x_i(t) \neq y_i(t) \Big\} \tag{4.3}$$

By Lemma 4.1, $D_{\neq}(t) \subset D_{\neq}(0) \subseteq I_0$ hence, if $i \in D_{\neq}(t)$, then $i \in \{1, .., n\}$. We denote by $|D_{\neq}(t)|$ the cardinality of $D_{\neq}(t)$ which thus counts the number of discrepancies at time t.

Lemma 4.3. *With the above notation, for any $t \geq 0$ we have*

$$\sum_{x=0}^{\epsilon^{-1}} |\xi_{\underline{x}(t),I(t)}(x) - \xi_{\underline{y}(t),J(t)}(x)| \leq |D_{\neq}(t)| + m \tag{4.4}$$

Proof. Shorthand $\xi_t(x) = \xi_{\underline{x}(t),I(t)}(x)$ and $\xi'_t(x) = \xi_{\underline{y}(t),J(t)}(x)$. Then

$$\begin{aligned}
\sum_{x=0}^{\epsilon^{-1}} |\xi_t(x) - \xi'_t(x)| &= \sum_{x=0}^{\epsilon^{-1}} \Big| \sum_{i \in I(t)} \mathbf{1}_{x_i(t)=x} - \sum_{i \in J(t)} \mathbf{1}_{y_i(t)=x} \Big| \\
&\leq \sum_{x=0}^{\epsilon^{-1}} \Big\{ \sum_{i \in D_{\neq}(t)} \big| \mathbf{1}_{x_i(t)=x} - \mathbf{1}_{y_i(t)=x} \big| + \sum_{i \in J(t) \setminus I(t)} \mathbf{1}_{y_i(t)=x} \Big\} \\
&= |D_{\neq}(t)| + |J(t) \setminus I(t)|
\end{aligned}$$

then the result follows since $J(t) \setminus I(t) \subseteq J_0 \setminus I_0$ and $|J_0 \setminus I_0| = m$. $\square$

Call $(\underline{x}^0(t), \underline{y}^0(t))$ the independent random walk process starting from $\underline{x}^0(0) = (x_1, .., x_n)$ and $\underline{y}^0(0) = (y_1, .., y_n)$. Call τ_i^0, $i = 1, .., n$, the first time t when $x_i^0(t) = y_i^0(t)$ and

$$D^0_{\neq}(t) = \{ i \in \{1, .., n\} : \tau_i^0 > t \} \tag{4.5}$$

and shall prove below that $|D^0_{\neq}(t)|$ stochastically bounds $|D_{\neq}(t)|$.

With this aim we introduce a process $(\underline{x}(t), I(t), \underline{y}(t), J(t), N(t); \underline{x}^0(t), \underline{y}^0(t))$ which couples the two processes $(\underline{x}(t), I(t), \underline{y}(t), J(t), N(t))$ and $(\underline{x}^0(t), \underline{y}^0(t))$. We denote its generator by

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_s + \hat{\mathcal{L}}_d + \hat{\mathcal{L}}_{\rm cr} + \hat{\mathcal{L}}_{\rm ann} + \hat{\mathcal{L}}^0 \tag{4.6}$$

$\hat{\mathcal{L}}_d$, $\hat{\mathcal{L}}_{\rm cr}$ and $\hat{\mathcal{L}}_{\rm ann}$ are the same as $\mathcal{L}_d$, $\mathcal{L}_{\rm cr}$ and $\mathcal{L}_{\rm ann}$ leaving unchanged $\underline{x}^0$ and $\underline{y}^0$. Also $\hat{\mathcal{L}}_s$ describes the same jumps as $\mathcal{L}_s$ but it also changes $\underline{x}^0$ and $\underline{y}^0$ with the following rules. For any $i \in I_0 \setminus I_{=}(t)$, if $x_i \to \min\{x_i + 1, \epsilon^{-1}\}$, then also $x_i^0 \to \min\{x_i^0 + 1, \epsilon^{-1}\}$ and, if $x_i \to \max\{x_i - 1, 0\}$, then also $x_i^0 \to \max\{x_i^0 - 1, 0\}$ (analogous rule for the y-jumps). The generator $\hat{\mathcal{L}}^0$ takes into account the independent jumps of x_i^0 and y_i^0 relative to the labels $i \in I_0 \cap I_{=}(t)$ which are not been taken into account by $\hat{\mathcal{L}}_s$. As before, for any $i \in I_0 \cap I_{=}(t)$, if $x_i, y_i \to \min\{x_i + 1, \epsilon^{-1}\}$, then also $x_i^0, y_i^0 \to \min\{x_i^0 + 1, \epsilon^{-1}\}$ and, if $x_i, y_i \to \max\{x_i - 1, 0\}$, then also $x_i^0, y_i^0 \to \max\{x_i^0 - 1, 0\}$.

Lemma 4.4. *If $\hat{P}$ is the law of the above process with generator $\hat{\mathcal{L}}$, then*

$$\hat{P}\big[D_{\neq}(t) \subset D^0_{\neq}(t)\big] = 1 \quad \text{for all } t \geq 0 \tag{4.7}$$

Proof. Let us consider $i \in \{1,..,n\}$ and suppose (for the sake of definiteness) that initially $x_i < y_i$ (recalling that $x_i^0 = x_i$ and $y_i^0 = y_i$). Call τ_i the first time t when either i leaves $I(t)$ or i enters into $I_=(t)$ We claim that $x_i(t) = x_i^0(t)$ and $y_i(t) \le y_i^0(t)$ for $t < \tau_i$ and since this implies (4.7) the claim will prove the lemma. Indeed the jumps described by $\hat{\mathcal{L}}_s$ preserve such a property and if $\hat{\mathcal{L}}_{\rm ann}$ involves the label i (in the case we are considering it will still be present after the jump event) then x_i is unchanged and y_i may only stay or decrease. □

As a direct consequence we have

Theorem 4.5. *There are positive constants c and b so that for any $t \ge 0$, any n, m and any initial configurations $\underline{x}$ and $\underline{y}$ as above*

$$\hat{E}\big[|D_{\neq}(t)|\big] \le cne^{-b\epsilon^2 t} \tag{4.8}$$

($\hat{E}$ denoting expectation with respect to the measure $\hat{P}$).

Proof. By (4.7) it is enough to prove the inequality for $E^0\big[|D^0_{\neq}(t)|\big]$, E^0 the expectation for the independent walkers process. The bound will follow from the inequality

$$p_t = p_t(i) := P^0[\tau_i^0 > t] \le ce^{-b\epsilon^2 t} \tag{4.9}$$

for any $i \in I_0 = \{1,\dots,n\}$. There is $\gamma > 0$ so that supposing $x_i < y_i$

$$p_{\epsilon^{-2}} \ge P^0\Big[x^0(\epsilon^{-2}) \ge \frac{y+x}{2},\; y^0(\epsilon^{-2}) \le \frac{y+x}{2}\Big] = \Big(P^0\Big[x^0(\epsilon^{-2}) \ge \frac{y+x}{2}\Big]\Big)^2 \ge \gamma$$

hence

$$p_t \le (1-\gamma)^{\epsilon^2 t - 1} = ce^{-b\epsilon^2 t}, \qquad b = -\log(1-\gamma), \qquad c = (1-\gamma)^{-1} \tag{4.10}$$

Now we have

$$\begin{aligned} P^0\big[|D_{\neq}(t)| = k\big] &= \sum_{\mathcal{I} \subseteq I_0 \colon |\mathcal{I}| = k} \quad \prod_{i \in \mathcal{I}} P^0\big[\tau_i^0 > t\big] \cdot \prod_{j \notin \mathcal{I}} P^0\big[\tau_j^0 \le t\big] \\ &= \binom{n}{k} p_t^k \, (1-p_t)^{n-k} \end{aligned} \tag{4.11}$$

then $E^0\big[|D_{\neq}(t)|\big] = n\, p_t$, this proves the Theorem. □

4.1 Convergence to linear profiles

We start by proving Theorem 2.4, to this end we show that two initial profiles with the same mass (or two initial configurations with the same total number of particles) become indistinguishable on the hydrodynamic time scale.

Proposition 4.6 (Loss of memory for ρ_t)**.** *Let $\rho_{\rm init}, \tilde{\rho}_{\rm init}$ be as in Definition 2.1. Suppose $F(0;\rho_{\rm init}) = F(0;\tilde{\rho}_{\rm init}) =: M$, then*

$$\lim_{t\to\infty} \sup_{r\in[0,1]} \big|F(r;S_t(\rho_{\rm init})) - F(r;S_t(\tilde{\rho}_{\rm init}))\big| = 0 \tag{4.12}$$

Proof. We shall use a corollary of Theorem 2.1 which may have an interest in its own right. Let ρ_{init}, ξ and $\rho_t(r)$ as in Theorem 2.1 then for any $t>0$

$$\lim_{\epsilon\to 0} E^{(\epsilon)}_{\xi}\Big[\max_{x\in[0,\epsilon^{-1}]} |\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t}) - F(\epsilon x;\rho_t)|\Big] = 0 \tag{4.13}$$

Proof of (4.13). For any $\zeta>0$ define

$$\mathcal{E}_\zeta(\epsilon,t) := \Big\{\max_{x\in[0,\epsilon^{-1}]}\big|F(\epsilon x;S_t(\rho_{\text{init}})) - \epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t})\big| \le \zeta\Big\}$$

Then, from the Cauchy-Schwarz inequality we have

$$\begin{aligned} &E^{(\epsilon)}_{\xi}\Big[\max_{x\in[0,\epsilon^{-1}]} |\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t}) - F(\epsilon x;\rho_t)|\Big] \\ &\le \zeta P^{(\epsilon)}_{\xi}\Big[\mathcal{E}_\zeta(\epsilon,t)\Big] + P^{(\epsilon)}_{\xi}\Big[\mathcal{E}_\zeta(\epsilon,t)^c\Big]^{1/2} E^{(\epsilon)}_{\xi}\Big[\Big(\max_{x\in[0,\epsilon^{-1}]} |\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t}) - F(\epsilon x;\rho_t)|\Big)^2\Big]^{1/2} \\ &\le \zeta + P^{(\epsilon)}_{\xi}\Big[\mathcal{E}_\zeta(\epsilon,t)^c\Big]^{1/2} E^{(\epsilon)}_{\xi}\Big[(\epsilon|\xi_{\epsilon^{-2}t}| + M)^2\Big]^{1/2} \end{aligned}$$

because $F(0,\rho_t) = F(0,\rho_{\text{init}}) = M$. By Theorem 2.1 $P^{(\epsilon)}_{\xi}\Big[\mathcal{E}_\zeta(\epsilon,t)^c\Big]$ vanishes while by Theorem 2.6, $E^{(\epsilon)}_{\xi}\Big[(M+\epsilon|\xi_{\epsilon^{-2}t}|)^2\Big] \le c$ uniformly in ϵ. (4.13) is thus proved.

Let $\{\tilde\xi\}$ be the family of initial data which approximate $\tilde\rho_{\text{init}}$, chosen in such a way that for all ϵ, $|\tilde\xi| = |\xi| =: n_\epsilon$. Calling $x_r := [\epsilon^{-1}r]$, $r\in[0,1]$, since $S_t(\rho_{\text{init}}), S_t(\tilde\rho_{\text{init}})$ are bounded we have

$$\begin{aligned} &\big|F(r;S_t(\rho_{\text{init}})) - F(r;S_t(\tilde\rho_{\text{init}}))\big| \le c\,|r-\epsilon x_r| + E^{(\epsilon)}_{\xi}\Big[|F(\epsilon x;S_t(\rho_{\text{init}})) - \epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t})|\Big] \\ &+E^{(\epsilon)}_{\tilde\xi}\Big[|F(\epsilon x;S_t(\tilde\rho_{\text{init}})) - \epsilon F_\epsilon(x;\tilde\xi_{\epsilon^{-2}t})|\Big] + \Big|E^{(\epsilon)}_{\xi}\Big[\epsilon F_\epsilon(x_r;\xi_{\epsilon^{-2}t})\Big] - E^{(\epsilon)}_{\tilde\xi}\Big[\epsilon F_\epsilon(x_r;\tilde\xi_{\epsilon^{-2}t})\Big]\Big| \end{aligned}$$

for some $c\ge 0$, then, by (4.13), (4.4) with $m=0$ and (4.8)

$$\begin{aligned} \Big|F(r;S_t(\rho_{\text{init}})) - F(r;S_t(\tilde\rho_{\text{init}}))\Big| &\le \lim_{\epsilon\to 0}\Big|E^{(\epsilon)}_{\xi}\Big[\epsilon F_\epsilon(x_r;\xi_{\epsilon^{-2}t})\Big] - E^{(\epsilon)}_{\tilde\xi}\Big[\epsilon F_\epsilon(x_r;\tilde\xi_{\epsilon^{-2}t})\Big]\Big| \\ &\le \lim_{\epsilon\to 0} c\epsilon n_\epsilon e^{-bt} \le cF(0;\rho_{\text{init}})e^{-bt} \end{aligned} \tag{4.14}$$

(4.12) is then proved. □

Proof of Theorem 2.4. Equation (2.13) follows from Theorem 2.3 and Proposition 4.6 with $\tilde\rho_{\text{init}} = \rho^{(M)}$. □

We fix arbitrary $M>0$ as an upper bound for the total macroscopic mass with $\epsilon^{-1}M$ bounding the total number of particles.

Definition 4.2. For any $\epsilon>0$ and any positive integer $N\le M\epsilon^{-1}$ we denote by $\eta^{(N,\epsilon)}\in\mathbb{N}^{\Lambda_\epsilon}$ the following particle approximation of the invariant profile $\rho^{(\epsilon N)}$. We set $\eta^{(N,\epsilon)}(\epsilon^{-1}) = 0$ and define iteratively for any $x\in[0,\epsilon^{-1}-1]$:

$$\sum_{y=0}^{x}\eta^{(N,\epsilon)}(y) = \Big\lceil \epsilon^{-1}\int_0^{\epsilon(x+1)}\rho^{(\epsilon N)}\Big\rceil \tag{4.15}$$

where $\lceil z\rceil$ is the smallest integer $\ge z$.

Observe that $\sum_{y=0}^{\epsilon^{-1}-1} \eta^{(N,\epsilon)}(y) = N$ and that for any $m > 0$ the sequence $\eta^{([\epsilon^{-1}m],\epsilon)}$ satisfies the conditions in Definition 2.1 with respect to $\rho_{\text{init}} = \rho^{(m)}$.

Proposition 4.7. *For any $\zeta > 0$ and $M > 0$ there are t and ϵ^* so that for any $\epsilon \le \epsilon^*$:*

$$\sup_{\xi:|\xi|\le M\epsilon^{-1}} E_\xi^{(\epsilon)}\Big[\max_{x\in[0,\epsilon^{-1}]} \big|\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t}) - F(\epsilon x;\rho^{(\epsilon|\xi|)})\big|\Big] \le \zeta \tag{4.16}$$

Proof. We split the interval $[0, M]$ into intervals of length θ, $\theta > 0$, calling $\theta_n = n\theta$. We choose θ so small that

$$\max_n \sup_{m\in[\theta_n,\theta_{n+1}]} \int_0^1 |\rho^{(\theta_n)}(r) - \rho^{(m)}(r)| \le \frac{\zeta}{2}$$

Let $\eta_t^{([\epsilon^{-1}\theta_n],\epsilon)}$ be the process with generator (1.1) and initial configuration $\eta^{([\epsilon^{-1}\theta_n],\epsilon)}$, then $|\epsilon|\eta^{([\epsilon^{-1}\theta_n],\epsilon)}| - \theta_n| \le \epsilon$. By (4.4) and Theorem 4.5 for any n and any ξ such that $\epsilon|\xi| \in [\theta_n, \theta_{n+1}]$,

$$\hat{E}\Big[\epsilon \sum_x \big|\eta_{\epsilon^{-2}t}^{([\epsilon^{-1}\theta_n],\epsilon)}(x) - \xi_{\epsilon^{-2}t}(x)\big|\Big] \le \theta + \epsilon + cMe^{-bt} \le \frac{\zeta}{4} \tag{4.17}$$

The last inequality requires t large enough: $cMe^{-bt} < \zeta/8$ and θ and ϵ small enough so that $\theta + \epsilon \le \zeta/8$. By Theorem 2.1 and (4.13), since $S_t(\rho^{(m)}) = \rho^{(m)}$, there is $\epsilon_1(t,\zeta;\theta)$ so that for all $\epsilon \le \epsilon_1(t,\zeta;\theta)$

$$\max_n \quad E_{\eta^{([\epsilon^{-1}\theta_n],\epsilon)}}^{(\epsilon)}\Big[\max_{x\in[0,\epsilon^{-1}]} \big|\epsilon F_\epsilon(x;\eta_{\epsilon^{-2}t}^{([\epsilon^{-1}\theta_n],\epsilon)}) - F(\epsilon x;\rho^{(\theta_n)})\big|\Big] \le \frac{\zeta}{4} \tag{4.18}$$

As a consequence

$$\begin{aligned}
&\sup_{\xi:\epsilon|\xi|\le M} E_\xi^{(\epsilon)}\Big[\max_{x\in[0,\epsilon^{-1}]} \big|\epsilon F_\epsilon(x;\xi_{\epsilon^{-2}t}) - F(\epsilon x;\rho^{(\epsilon|\xi|)})\big|\Big] \\
&\qquad \le \sup_{\xi:\epsilon|\xi|\le M} E_\xi^{(\epsilon)}\Big[\max_{x\in[0,\epsilon^{-1}]} \big|\epsilon F_\epsilon(x;\eta_{\epsilon^{-2}t}^{([\epsilon^{-1}\theta_n],\epsilon)}) - F(\epsilon x;\rho^{(\epsilon|\xi|)})\big|\Big] + \frac{\zeta}{4} \\
&\qquad \le \frac{\zeta}{2} + \max_n \sup_{m\in[\theta_n,\theta_{n+1}]} \int_0^1 |\rho^{(\theta_n)}(r) - \rho^{(m)}(r)| < \zeta
\end{aligned}$$

this concludes the proof. □

4.2 Evolving profiles

Proof of Theorem 2.6. From the definition of the generator (1.1) we infer that the induced process $|\xi_t|$ counting the number of particles at time t evolves with the generator

$$\mathcal{L}^{(\epsilon)} f(|\xi|) = j\epsilon\Big\{\Big(f(|\xi|+1) - f(|\xi|)\Big) + \mathbf{1}_{|\xi|>0}\Big(f(|\xi|-1) - f(|\xi|)\Big)\Big\} \tag{4.19}$$

acting on bounded functions $f : \mathbb{N} \to \mathbb{R}$. Such generator is immediately recognized to be the generator of the continuous time symmetric random walk on $\mathbb{N}$ at rate $j\epsilon$ and reflected at the origin. $\square$

We also have that calling $\mathcal{P}_x^{(\epsilon)}$, $x \in \mathbb{N}$, the law of the random walk x_t with generator $\mathcal{L}^{(\epsilon)}$ starting from x:

Lemma 4.8. *Let $M' > 0$ and $T > 0$ then for any $\delta > 0$ there is M so that for all ϵ small enough, any $x \leq \epsilon^{-1} M'$ and any $t \leq \epsilon^{-3} T$,*

$$\sup_{t \leq \epsilon^{-2} T} \mathcal{P}_x^{(\epsilon)} \left[|x_t - x| \leq \delta \right] \geq 1 - \delta, \qquad \sup_{t \leq \epsilon^{-3} T} \mathcal{P}_x^{(\epsilon)} \left[x_t \geq \epsilon^{-1} M \right] \leq \delta \tag{4.20}$$

Proof of Theorem 2.5. The last statement of the theorem, i.e. that

$$M_t^{(\epsilon)} := \epsilon \left| \xi_{\epsilon^{-3} t} \right| \to B_{jt} \qquad \text{as } \epsilon \to 0 \qquad \textit{in law} \tag{4.21}$$

with $(B_t)_{t \geq 0}$ the brownian motion on $\mathbb{R}_+$ with reflection at the origin, starting from $B_0 = \lim_{\epsilon \to 0} M_0^{(\epsilon)} = \lim_{\epsilon \to 0} \epsilon |\xi|$, follows from Theorem 2.6 and the fact that the diffusive scaling limit of the random walk is Brownian motion.

- Subcritical regime. (2.15) follows directly from (4.22).
- Critical regime. Let $t^* = \epsilon^{-3} t - s$, then by Lemma 4.8 for any given $s > 0$, with probability $\geq 1 - \delta$, $|\xi_{t^*} \leq \epsilon^{-1} M$. By (4.22), choosing s large enough in the set $|\xi_{t^*}| \leq \epsilon^{-1} M$,

$$E_{\xi_{t^*}}^{(\epsilon)} \left[\max_x |\epsilon F_\epsilon(x; \xi_{\epsilon^{-2} s}) - F(\epsilon x; \rho^{(\epsilon |\xi_{t^*}|)})| \right] \leq \frac{\zeta}{2}$$

On the other hand by (4.20) for ϵ small enough

$$P_{\xi_{t^*}}^{(\epsilon)} \left[||\xi_{t^*}| - |\xi_{t^* + \epsilon^{-2} s}|| \leq \delta \right] \geq 1 - \delta$$

so that (2.16) follows from the continuity in m of $F(0; \rho^{(m)})$. $\square$

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