Duality for stochastic models of transport

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Abstract

We study three classes of continuous time Markov processes (inclusion process, exclusion process, independent walkers) and a family of interacting diffusions (Brownian energy process). For each model we define a boundary driven process which is obtained by placing the system in contact with proper reservoirs, working at different particle densities or different temperatures. We show that all the models are exactly solvable by duality, using a dual process with absorbing boundaries. The solution does also apply to the so-called thermalization limit in which particles or energy is instantaneously redistributed among sites.

The results shows that duality is a versatile tool for analyzing stochastic models of transport, while the analysis in the literature has been so far limited to particular instances. Long-range correlations naturally emerge as a result of the interaction of dual particles at the microscopic level and the explicit computations of covariances match, in the scaling limit, the predictions of the macroscopic fluctuation theory.

1 Introduction

Interacting particle systems are classical models to study non-equilibrium statistical mechanics. The standard setting is the one in which a system is placed in contact with reservoirs working at different parameters that create a stationary state characterized by a non-zero averaged current. The prototypical example are the Symmetric Exclusion process with at most one particle per site connected to birth and death process at the boundaries [L, D] and the KMP process [KMP] connected to reservoirs which impose at the boundaries Boltzmann-Gibbs distribution with different temperatures. The Symmetric exclusion process is a model for transport of a discrete quantity, whereas the KMP process models transport of a continuous quantity.

Problems that are very hard for classical Hamiltonian systems – for instance deriving Fourier law starting from the microscopic evolution – can be successfully approached using stochastic models. Furthermore stochastic models of transport have been used to prove new theorems in non-equilibrium statistical mechanics, such as the fluctuation theorem [GC, K], to introduce new principles, such as the additivity principle [BD], to construct new schemes, such as the macroscopic fluctuation theory that describes the density and current large deviations for diffusive systems [BDGJL1, DLS], to test new algorithms, such as cloning algorithms to simulate rare events ([GKLT]). Recently, the connection between deterministic Hamiltonian systems and stochastic models is emerging either by considering evolutions in which they are coupled [BO] or by considering slow/fast variables [DL].

In the analysis of general stochastic processes a powerful tool, when available, is duality theory [S, L]. In the duality approach, the original process is studied via a simpler process, called *dual process*. This has been applied in different contexts, including interacting particles systems, interacting diffusions, queueing theory and mathematical population genetics (for a recent review on duality, which also include many references, see [JK]). For stochastic models of transport, *in the presence of boundary reservoirs*, duality has been found originally for the boundary driven Symmetric Exclusion process with at most one particle per site [Spo] and for the KMP model [KMP]. Recently duality has been obtained also for other models, such as Symmetric Inclusion process and for interacting diffusion introduced to model heat conduction [GKR, GRV]. Typically the duality approach allows to study n-point correlation functions in the original process with L degrees of freedom using n interacting dual particles (with $n << L$). Thus the problem of studying correlations functions in a large (eventually infinite) systems is reduced to a finite dimensional problem. In all those cases the existence of a dual process has been related to the study of hidden symmetries and/or change of representation of an abstract operator [GKRV, SS].
In this paper we define classes of boundary driven interacting particle systems and a family of interacting diffusions with reservoirs. The particles systems interpolate from the Inclusion model with a Negative-Binomial stationary measure at equilibrium, to the Exclusion model with a Binomial equilibrium state, via Independent walkers with a Poisson stationary measure. The interacting diffusions are given by the so-called Brownian Energy process, having a Gamma distribution at equilibrium.
For all the models we construct a dual process which allows to study correlation functions in the stationary non-equilibrium state (i.e. different reservoir densities or temperatures) using the absorption probabilities of dual walkers at the boundaries. The results include, but are not limited to, some of the models mentioned above. The results hold true for any value of the reservoir parameters. This shows that duality is a versatile tool in the analysis of such systems.
The KMP model is not included in the classes of model we consider in a first stage. However it can be recovered by considering an instantaneous thermalization limit of one element of the family of the Brownian Energy model, as already remarked in [GKRV]. We construct here the thermalization limit for all the models, obtaining both in the discrete and in the continuous case, several redistribution rules of particles/energy. A non-trivial stationary state is found in this case even considering only one site, since the reservoirs are not additive.
Finally, we compute in this paper the explicit microscopic expressions for the averages and covariances for many parameters values of our models in non-equilibrium setting. From the analysis, we deduce that the macroscopic limit of the microscopic expressions are in full agreement with the predictions of the macroscopic fluctuation theory.
The paper is organized as follows. In the next section we introduce the models and discuss the relation between them, including scaling limits. We describe the equilibrium states in section 3, both in the case of free boundary conditions and in the case of systems connected to reservoirs. Section 4 presents the duality result and describes the dual processes, together with the illustration of the use of duality in the non-equilibrium setting. As an example, the linear temperature profile are computed for all the models, and the stationary measure of boundary driven independent walkers is deduced from duality. Section 5 describes the instantaneous thermalization limit of our models, and their duals. The last two section deal with correlations in the stationary non equilibrium state (section 6) and the comparison to the predictions of the macroscopic fluctuation theory (section 7).

2 Models definition

In this section we introduce our models. In the most complete setting, they are constituted by a bulk which is kept in a non equilibrium state by the contact with particles or energy reservoirs. In particular, we consider one-dimensional systems on a finite lattice $\{1, \ldots, L\}$, whose boundaries (i.e. sites 1 and L) interact with the reservoirs. When needed, the reservoirs themselves will be represented

by two extra sites, namely sites 0 and $L+1$.
Accordingly, the generators of the random processes associated with our models can be generically expressed as the sum of three terms

$$\mathcal{L} = \mathcal{L}_a + \mathcal{L}_0 + \mathcal{L}_b \ , \tag{2.1}$$

where $\mathcal{L}_0$ represents the generator of the dynamics in the bulk, while $\mathcal{L}_a$ and $\mathcal{L}_b$ represent the generators of the reservoirs.
We will consider four models: three classes of interacting particle systems, characterized by the different interactions between the particles, and one family of interacting diffusions introduced to model heat conduction [GKR, GKRV, GRV]. The models are:

1. the Symmetric Inclusion Process (SIP), with *attractive* interaction between neighboring particles;
2. the Symmetric Exclusion Process (SEP), with *repulsive* interaction between neighboring particles;
3. the Independent Random Walkers (IRW), without interactions among particles;
4. the Brownian Energy Process (BEP).

In the first three cases the dynamic variable is a vector that specifies the number of particle on each site: $\eta = (\eta_1, ..., \eta_L) \in \Omega$; here Ω, the state space, depends on the model and will be defined ahead. In the case of the BEP the dynamic variable is a vector z representing the energies on each site of the lattice: $z = (z_1, \dots, z_L) \in \Omega \equiv \mathbb{R}_+^L$.

2.1 Interacting particle systems

The generators of the reservoirs for SIP, SEP and IRW have the following general form:

$$\mathcal{L}_a f(\eta) = b(\eta_1)[f(\eta^{0,1}) - f(\eta)] + d(\eta_1)[f(\eta^{1,0}) - f(\eta)], \tag{2.2}$$

$$\mathcal{L}_b f(\eta) = b(\eta_L)[f(\eta^{L+1,L}) - f(\eta)] + d(\eta_L)[f(\eta^{L,L+1}) - f(\eta)]. \tag{2.3}$$

According to (2.2) and (2.3) particles are injected into the system through the boundaries with rate $b(n)$ with $n \in \mathbb{N}_0$, and removed from the same sites with rate $d(n)$. While $b(n)$ is model-dependent, the annihilation rate is not, being in any case proportional to the number of particles at the boundary site.
We introduce now our models by defining the actions of the generators $\mathcal{L}$ on the functions $f : \Omega \to \mathbb{R}$. Hereafter, we will use the symbol $\eta^{i,i+1}$ to denote the configuration obtained from η by moving a particle from site i to site $i+1$, i.e. $\eta^{i,i+1} = (\eta_1, \dots, \eta_i - 1, \eta_{i+1} + 1, \dots, \eta_L)$.

Inclusion walkers SIP($2k$). This is an inclusion process in which each site can accomodate an arbitrary number of particles, thus $\Omega = \mathbb{N}_0^L$. In the bulk each particle may jump to its left or right neighbouring site with rates proportional to the number of particles in the departure site and to the number of particles in the arrival site. In each boundary site particles are created with a rate proportional to $2k$ plus the number of particles sitting in that site; $k \in \mathbb{R}_+$ labels the class of models. The generator is

$$\begin{aligned} \mathcal{L}^{SIP} f(\eta) &= \mathcal{L}_a^{SIP} f(\eta) + \mathcal{L}_0^{SIP} f(\eta) + \mathcal{L}_b^{SIP} f(\eta) \\ &= \alpha(2k+\eta_1)\left[f(\eta^{0,1}) - f(\eta)\right] + \gamma\eta_1\left[f(\eta^{1,0}) - f(\eta)\right] \\ &+ \sum_{i=1}^{L-1} \eta_i(2k+\eta_{i+1})\left[f(\eta^{i,i+1}) - f(\eta)\right] + \eta_{i+1}(2k+\eta_i)\left[f(\eta^{i+1,i}) - f(\eta)\right] \\ &+ \delta(2k+\eta_L)\left[f(\eta^{L+1,L}) - f(\eta)\right] + \beta\eta_L\left[f(\eta^{L,L+1}) - f(\eta)\right]. \end{aligned} \tag{2.4}$$

The positive numbers α and γ (resp. δ and β) tune the creation and annihilation rates of the left (resp. right) reservoirs.

Exclusion walkers SEP($2j$). For this process the maximum occupation number at each site is $2j \in \mathbb{N}$, thus $\Omega = \{0, 1, \ldots, 2j\}^L$. In the bulk particles jump independently to nearest neighbouring lattices sites at rate proportional to the number of particles in the departure site times the number of holes in the arrival site. The reservoirs inject particles in the systems with a rate proportional to the holes in the boundary sites. The generator is

$$\begin{aligned}
\mathcal{L}^{SEP} f(\eta) &= \mathcal{L}_a^{SEP} f(\eta) + \mathcal{L}_0^{SEP} f(\eta) + \mathcal{L}_b^{SEP} f(\eta) \\
&= \alpha(2j - \eta_1)\left[f(\eta^{0,1}) - f(\eta)\right] + \gamma\eta_1\left[f(\eta^{1,0}) - f(\eta)\right] \\
&+ \sum_{i=1}^{L-1} \eta_i(2j - \eta_{i+1})\left[f(\eta^{i,i+1}) - f(\eta)\right] + \eta_{i+1}(2j - \eta_i)\left[f(\eta^{i+1,i}) - f(\eta)\right] \\
&+ \delta(2j - \eta_L)\left[f(\eta^{L+1,L}) - f(\eta)\right] + \beta\eta_L\left[f(\eta^{L,L+1}) - f(\eta)\right] .
\end{aligned} \tag{2.5}$$

The parameters α, γ, δ, β have the same meaning as in the SIP($2k$).

Independent random walkers IRW. In this case each particle jumps independently to nearest neighbouring lattices sites at rate 1, and each site can accomodate an arbitrary number of particles, thus $\Omega = \mathbb{N}_0^L$. Jumps occur with the same probability to the right and to the left, while particles are created at rates α and δ irrespective of the numer of particles at the boundaries. Therefore the system is described by the generator

$$\begin{aligned}
\mathcal{L}^{IRW} f(\eta) &= \mathcal{L}_a^{IRW} f(\eta) + \mathcal{L}_0^{IRW} f(\eta) + \mathcal{L}_b^{IRW} f(\eta) \\
&= \alpha\left[f(\eta^{0,1}) - f(\eta)\right] + \gamma\eta_1\left[f(\eta^{1,0}) - f(\eta)\right] \\
&+ \sum_{i=1}^{L-1} \eta_i\left[f(\eta^{i,i+1}) - f(\eta)\right] + \eta_{i+1}\left[f(\eta^{i+1,i}) - f(\eta)\right] \\
&+ \delta\left[f(\eta^{L+1,L}) - f(\eta)\right] + \beta\eta_L\left[f(\eta^{L,L+1}) - f(\eta)\right] .
\end{aligned} \tag{2.6}$$

The dynamics in the bulk can be further described by saying that if at site i there are η_i particles, one of the particle jumps at rate η_i either to the left or to the right. As in the previous cases, parameters γ and β define the annihilation processes.

Remark 2.1. *The effect of the reservoirs is to impose the average number of particles on the left and on the right sides of the chains. With some misuse of language, but sticking to standard notations, we will call "densities" these averages and we will denote them ρ_a (left reservoir) and ρ_b (right reservoir). The vaules of ρ_a and ρ_b are reported in the table below and computed in Sec.3.*

System	ρ_a	ρ_b
SIP	$2k\frac{\alpha}{\gamma-\alpha}$	$2k\frac{\delta}{\beta-\delta}$
SEP	$2j\frac{\alpha}{\gamma+\alpha}$	$2j\frac{\delta}{\beta+\delta}$
IRW	$\frac{\alpha}{\gamma}$	$\frac{\delta}{\beta}$

Table 1: Definition of ρ_a and ρ_b.

Remark 2.2. *Note that the SIP process requires $\gamma > \alpha$ and $\beta > \delta$. This condition turns out to be necessary in order for the system to reach a stationary state (see also formula* (3.8)*).*

Remark 2.3. *It is interesting to remark that the exclusion (resp. inclusion) walkers with parameters* $(\alpha, \gamma', \delta, \beta')$ *converges to the independent walkers with parameters* $(\alpha, \gamma, \delta, \beta)$ *in the limit* $j \to \infty$ *(resp.* $k \to \infty$*) under the scaling* $\gamma' = 2j\gamma$*,* $\beta' = 2j\beta$ *(resp.* $\gamma' = 2k\gamma$*,* $\beta' = 2k\beta$*). Indeed, in this limit the generators* $\frac{\mathcal{L}^{SIP}}{2j}$ *and* $\frac{\mathcal{L}^{SEP}}{2j}$ *converge to* $\mathcal{L}^{IRW}$*. This remark can be put on rigorous grounds by using the Trotter-Kunz theorem (see Theorem 2.12 of [L]); see for istance [GKRV] for the proof in the case of SEP(*$2j$*).*

2.2 Interacting diffusions

The last process we consider, the Brownian Energy Process, was originally introduced in [GKRV] and recognized as dual of a system of inclusion random walkers. Here we present the boudary driven version of the BEP.

Brownian energy process BEP($2k$**).** This model describes symmetric energy exchange between nearest neighbouring sites. The dynamical variables (energies) are collected in the vector $z = (z_1, \ldots, z_L) \in \mathbb{R}_+^L$ and the generator is

$$\begin{aligned}
\mathcal{L}^{BEP} &= \mathcal{L}_a^{BEP} f(\eta) + \mathcal{L}_0^{BEP} f(\eta) + \mathcal{L}_b^{BEP} f(\eta) \\
&= T_a \left(2k \frac{\partial}{\partial z_1} + z_1 \frac{\partial^2}{\partial z_1^2} \right) - \frac{1}{2} z_1 \frac{\partial}{\partial z_1} \\
&+ \sum_{i=1}^{L-1} z_i z_{i+1} \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2 - 2k(z_i - z_{i+1}) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right) \\
&+ T_b \left(2k \frac{\partial}{\partial z_L} + z_L \frac{\partial^2}{\partial z_L^2} \right) - \frac{1}{2} z_L \frac{\partial}{\partial z_L} \; .
\end{aligned} \tag{2.7}$$

Remark 2.4. *The origin of the bulk dynamics, generated by*

$$\mathcal{L}_0^{BEP} f(z) = \sum_{i=1}^{L-1} z_i z_{i+1} \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2 - 2k(z_i - z_{i+1}) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right) \tag{2.8}$$

can be explained as follows [GKR, GKRV]. Consider $m = 4k \in \mathbb{N}$ *velocity variables on each site* i *and call them* $v_{i,\alpha}$ *with* $\alpha = 1, \ldots, m$*. Suppose that they evolve with the following generator*

$$\mathcal{L}_0^{BMP} f(v) = \sum_{i=1}^{L-1} \sum_{\alpha,\beta=1}^{m} \left(v_{i,\alpha} \frac{\partial}{\partial v_{i+1,\beta}} - v_{i+1,\beta} \frac{\partial}{\partial v_{i,\alpha}} \right)^2 \tag{2.9}$$

which defines a process, called Brownian Momentum Process, introduced in [BO, GK]. Each term in $\mathcal{L}_0^{BMP}$ *represents a rotation in the plane* $(v_{i,\alpha}, v_{i+1,\beta})$*, therefore it conserves the total length* $v_{i,\alpha}^2 + v_{i+1,\beta}^2$*, i.e. the total kinetic energy. One can check that the BEP(*$2k$*) is the evolution process, induced by (2.9), of the total energies on each site*

$$z_i = \sum_{\alpha=1}^{m} v_{i,\alpha}^2 \; . \tag{2.10}$$

The generator of the BEP reservoirs $\mathcal{L}_a^{BEP}$ *and* $\mathcal{L}_b^{BEP}$*, that will be discussed in some details in Sec. 3, impose an average energy* $4kT_a$ *on the left, and an average energy* $4kT_b$ *on the right. The choice of their form is motivated as follows. Consider an Ornestein-Uhlenbeck process on each of the* m *velocities at site* 1 *of the Brownian Momentum process (2.9), namely*

$$\mathcal{L}_a^{BMP} = \sum_{\alpha=1}^{m} 2T \frac{\partial^2}{\partial v_{1,\alpha}^2} - v_{1,\alpha} \frac{\partial}{\partial v_{1,\alpha}} \; . \tag{2.11}$$

Since in the stationary state of this reservoir the $\{v_{1,\alpha}\}_{\alpha=1,\dots,m}$ are independent centered Gaussian with variance T then, using (2.10), the expectation of z_1 is $\mathbb{E}(z_1) = \sum_{\alpha=1}^{m} \mathbb{E}(v_{1,\alpha}^2) = mT = 4kT$.

2.3 Scaling limit of the particle systems

Besides duality, there is another relation connecting the bulk part of the BEP with generator $\mathcal{L}_0^{BEP}$ (2.8), and the bulk part of the SIP with generator $\mathcal{L}_0^{SIP}$ (third line in (2.4)). The BEP can be indeed obtained from the SIP, through a suitable scaling limit, by a reinterpretation of this process as a model of energy transport, by supposing that each particle carries a quantum of energy ϵ. In this interpretation, since $\mathcal{L}_0^{SIP}$ conserves the number of particles, then it conserves the total energy. Consider the free boundary inclusion process $\eta(t) = (\eta_1(t), \dots, \eta_L(t))$ generated by $\mathcal{L}_0^{SIP}$ and let N be the total number of particles, i.e. $N = \sum_{i=1}^{L} \eta_i$. Let ϵ be a parameter of the order of $1/N$, then one expects η_i to be of the order of ϵ^{-1} as $\epsilon \to 0$ (despite attractive interactions for any finite k there are no condensation phenomena in the SIP; one needs to rescale k with ϵ to see particles coalescing into a single site; see [GRV2]). Then one may investigate the continuous dynamics generated in the limit as $\epsilon \to 0$ on the variables $z_i(t) = \epsilon\eta_i(t)$. It turns out that the limiting dynamics for $z(t)$ is generated by $\mathcal{L}_0^{BEP}$.

Proposition 2.5. *Let $\eta(t) = (\eta_1(t), \dots, \eta_L(t))$ be the bulk inclusion process generated by $\mathcal{L}_0^{SIP}$ with N particles. Let $\epsilon = \mathcal{E}/N$ for some fixed $\mathcal{E} > 0$. Then the process $z(t) = (z_1(t), \dots, z_L(t))$ where $z_i(t) = \epsilon\, \eta_i(t)$ is, in the limit $\epsilon \to 0$, the bulk Brownian energy process generated by $\mathcal{L}_0^{BEP}$ with total energy $\mathcal{E}$.*

Proof. Let $F : \mathbb{R}_+^L \to \mathbb{R}$, $F = F(z)$ be a two times countinuously differentiable function, i.e. $F \in \mathcal{C}^2(\mathbb{R}_+^L)$. Let $z^\epsilon = (z_1^\epsilon, \dots, z_L^\epsilon) \in \mathbb{R}_+^L$ be such that $z^\epsilon/\epsilon \in \mathbb{N}_+^L$, then for any F as above, there exists $f : \mathbb{N}_0^L \to \mathbb{R}$, $f = f(\eta)$, such that

$$F\left(z_1^\epsilon, \dots, z_L^\epsilon\right) := f\left(\frac{z_1^\epsilon}{\epsilon}, \dots, \frac{z_L^\epsilon}{\epsilon}\right) . \tag{2.12}$$

Let $\mathcal{L}_0^\epsilon$ be the generator of the process $z^\epsilon(t)$ induced by the SIP, then $\mathcal{L}_0^\epsilon$ acts on $F = F(z^\epsilon)$ as follows:

$$\begin{aligned}
\left[\mathcal{L}_0^\epsilon F\right](z^\epsilon) &= \left[\mathcal{L}_0^{SIP} f\right]\left(\frac{z^\epsilon}{\epsilon}\right) \\
&= \sum_{i=1}^{L-1} \Bigg\{ \frac{z_i^\epsilon}{\epsilon}\left(2k + \frac{z_{i+1}^\epsilon}{\epsilon}\right)\left[f\left(\frac{z_1^\epsilon}{\epsilon}, \dots, \frac{z_i^\epsilon}{\epsilon} - 1, \frac{z_{i+1}^\epsilon}{\epsilon} + 1, \dots, \frac{z_L^\epsilon}{\epsilon}\right) - f\left(\frac{z^\epsilon}{\epsilon}\right)\right] \\
&\qquad + \frac{z_{i+1}^\epsilon}{\epsilon}\left(2k + \frac{z_i^\epsilon}{\epsilon}\right)\left[f\left(\frac{z_1^\epsilon}{\epsilon}, \dots, \frac{z_i^\epsilon}{\epsilon} + 1, \frac{z_{i+1}^\epsilon}{\epsilon} - 1, \dots, \frac{z_L^\epsilon}{\epsilon}\right) - f\left(\frac{z^\epsilon}{\epsilon}\right)\right] \Bigg\} \\
&= \sum_{i=1}^{L-1} \Bigg\{ \frac{z_i^\epsilon}{\epsilon}\left(2k + \frac{z_{i+1}^\epsilon}{\epsilon}\right)\left[F\left(z_1^\epsilon, \dots, z_i^\epsilon - \epsilon, z_{i+1}^\epsilon + \epsilon, \dots, z_L^\epsilon\right) - F\left(z^\epsilon\right)\right] \\
&\qquad + \frac{z_{i+1}^\epsilon}{\epsilon}\left(2k + \frac{z_i^\epsilon}{\epsilon}\right)\left[F\left(z_1^\epsilon, \dots, z_i^\epsilon + \epsilon, z_{i+1}^\epsilon - \epsilon, \dots, z_L^\epsilon\right) - F\left(z^\epsilon\right)\right] \Bigg\} .
\end{aligned}$$

Suppose that z^ϵ converges to a finite limit $z^\epsilon \to z \in \mathbb{R}_+^L$ as $\epsilon \to 0$. Then, from the regularity assumptions on F, we have

$$\begin{aligned}
[\Delta_{i,i+1}^\epsilon F](z^\epsilon) &:= F(z_1^\epsilon, \dots, z_{i-1}^\epsilon - \epsilon, z_i^\epsilon + \epsilon, \dots, z_L^\epsilon) - F(z_1^\epsilon, \dots, z_{i-1}^\epsilon, z_i^\epsilon, \dots, z_L^\epsilon) \\
&= -\epsilon\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right) F(z^\epsilon) + o(\epsilon) \ ,
\end{aligned} \tag{2.13}$$

while

$$\begin{aligned}[\Delta^\epsilon_{i+1,i}F](z^\epsilon) &:= F(z_1^\epsilon,\ldots,z_{i-1}^\epsilon+\epsilon, z_i^\epsilon-\epsilon,\ldots,z_L^\epsilon) - F(z_1^\epsilon,\ldots,z_{i-1}^\epsilon, z_i^\epsilon,\ldots,z_L^\epsilon) \\ &= \epsilon\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)F(z^\epsilon) + o(\epsilon)\ , \end{aligned} \tag{2.14}$$

and

$$[(\Delta^\epsilon_{i+1,i}+\Delta^\epsilon_{i,i+1})F](z^\epsilon) = \epsilon^2\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)^2 F(z^\epsilon) + o(\epsilon^2)\ .$$

Therefore we have

$$[\mathcal{L}_0^\epsilon F](z^\epsilon) = \left[-2k(z_i^\epsilon - z_{i+1}^\epsilon)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right) + z_i^\epsilon z_{i+1}^\epsilon\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)^2\right] F(z^\epsilon) + o(1)\ . \tag{2.15}$$

Thus, for any F as above, $\lim_{\epsilon\to 0}[\mathcal{L}_0^\epsilon F](z^\epsilon) = [\mathcal{L}_0^{BEP}F](z)$. Moreover the total energy is clearly conserved in the limit and it is given by $\sum_{i=1}^L z_i = \sum_{i=1}^L z_i^\epsilon = \epsilon\, N = \mathcal{E}$. □

The same scaling analysis of the inclusion walkers can be performed on the bulk dynamics of independent random walkers. This yields a deterministic process as scaling limit, which is also dual to independent random walkers (cfr. [GKRV]).

Proposition 2.6. *Let $\eta(t) = (\eta_1(t),\ldots,\eta_L(t))$ be the bulk process generated by $\mathcal{L}_0^{IRW}$ with N particles. Let $\epsilon = \mathcal{E}/N$ for some fixed $\mathcal{E} > 0$. Then the process $y(t) = (y_1(t),\ldots,y_L(t))$ where $y_i(t) = \epsilon\,\eta_i(t)$ is, in the limit $\epsilon \to 0$, the deterministic energy process (DEP) with total energy $\sum_{i=1}^{L-1} y_i(t) = \mathcal{E}$ generated by*

$$\mathcal{L}_0^{DEP} = \sum_{i=1}^{L-1}(y_i - y_{i+1})\left(\frac{\partial}{\partial y_{i+1}} - \frac{\partial}{\partial y_i}\right)\ .$$

Remark 2.7. *One may wonder whether there exists a diffusion process arising as a limit of the Exclusion process. By performing an analogous scaling as above, the rates of the SEP take the form $Nz_i(2j - Nz_{i+1})$ that become negative in the limit as $N \to \infty$. Consistently the limit of the SEP generator is a second order differential operator that cannot be interpreted as the generator of a Markov process, since it is has a negative coefficient in front of the second order derivatives, i.e.*

$$\sum_{i=1}^{L} -z_i z_{i+1}\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right)^2 - 2j(z_i - z_{i+1})\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)\ . \tag{2.16}$$

Remark 2.8. *The same scaling limit which transfoms the bulk dynamics of the SIP into the one of the BEP does not work with the reservoirs. Indeed, applying to $\mathcal{L}_a^{SEP}$ the scaling of Propositon 2.5, the resulting generator is*

$$(\alpha-\gamma)z_1\frac{\partial}{\partial z_1} \tag{2.17}$$

which produces a deterministic behavior: $z_1(t) = z_1(0)e^{(\alpha-\gamma)t}$. On the other hand it is simple to check that the thermal bath of the BEP can be obtained from a boundary driven SIP with a modified reservoir generated by

$$\mathcal{L}_{a,q}^{SIP} = \left(2kq + \left(q - \frac{1}{2}\right)\eta_1\right)\left[f(\eta^{0,1}) - f(\eta)\right] + q\eta_1\left[f(\eta^{1,0}) - f(\eta)\right] \tag{2.18}$$

with the condition $q\epsilon \to T_a$ as $\epsilon \to 0$.

3 Stationary measures at equilibrium

The models introduced in the previous section are Markov processes with discrete or continuous state spaces. The long term behaviour of the processes are described by their stationary measures. In general it is hard to determine such measures and, in fact, the invariant states of SIP, SEP and BEP in non-equilibrium conditions are not explicitly known. The problem of finding the explicit form of the invariant states is greatly simplyfied at equilibrium. The equilibrium condition for our systems can be obtained in two ways: either by suppressing the reservoirs (i.e. considering only the bulk dynamics $\mathcal{L}_0$) or, retaining the reservoirs, by imposing equal densities or equal temperatures at the boundaries of the chain, i.e. $\rho = \rho_a = \rho_b$ or $T = T_a = T_b$.
In the first case there exists an infinite family of reversible measures labelled by a continuous parameter. In the second case (i.e. in the presence of the reservoir) at density ρ (resp. at temperature T) the boundary conditions select one reversible measure.

3.1 Equilibrium product measures

Reversible invariant probability measure $\mathbb{P}$ of the bulk dynamics generated by $\mathcal{L}_0$ can be obtained by imposing the detailed balance condition. When the state space Ω is finite or countable, this condition is expressed by requiring that for any pair of configurations η, $\eta' \in \Omega$ the probability $\mathbb{P}$ satisfies

$$L_0(\eta,\eta')\mathbb{P}(\eta) = L_0(\eta',\eta)\mathbb{P}(\eta') \tag{3.1}$$

where $L_0(\eta,\eta')$ is the transition rate from the configuration η to η', i.e. $L_0(\eta,\eta') = \mathcal{L}_0 f(\eta)$ with $f(\eta) = \delta_{\eta,\eta'}$. When the state space Ω is continuos, a probability measure with density $\psi(x)$ is said to be reversible stationary measure if, for all functions f and g in the domain of the generator $\mathcal{L}_0$, it holds

$$\int f(x)\mathcal{L}_0 g(x)\psi(x)dx = \int \mathcal{L}_0 f(x) g(x)\psi(x)dx \ . \tag{3.2}$$

By imposing (3.1) in the case of SIP, SEP, IRW and (3.2) in the case of BEP and requiring the factorization of the probability measure one obtain the reversible measures described in the following proposition, whose proof is left to the reader.

Proposition 3.1. *For the bulk processes with generator $\mathcal{L}_0$ defined in Sec. 2 we have*

Inclusion walkers SIP$(2k)$
The process with generator $\mathcal{L}_0^{SIP}$ has a reversible stationary measure given by products of generalized `Negative Binomial` *measures with parameters $2k > 0$ and arbitrary $0 < p < 1$, i.e.*

$$\mathbb{P}(\eta) = \prod_{i=1}^{L} \frac{p^{\eta_i}}{\eta_i!}\frac{\Gamma(2k+\eta_i)}{\Gamma(2k)}(1-p)^{2k} \ . \tag{3.3}$$

Exclusion walkers SEP$(2j)$
The process with generator $\mathcal{L}_0^{SEP}$ has reversible stationary measure given by products of `Binomial` *measures with parameters $2j \in \mathbb{N}$ and arbitrary $0 < p < 1$, i.e.*

$$\mathbb{P}(\eta) = \prod_{i=1}^{L} \frac{\left(\frac{p}{1-p}\right)^{\eta_i}}{\eta_i!}\frac{\Gamma(2j+1)}{\Gamma(2j+1-\eta_i)}(1-p)^{2j} \ . \tag{3.4}$$

Independent random walkers IRW
The process with generator $\mathcal{L}_0^{IRW}$ has reversible stationary measure given by products of `Poisson` *distribution with arbitrary parameter $\lambda > 0$ i.e.*

$$\mathbb{P}(\eta) = \prod_{i=1}^{L} \frac{\lambda^{\eta_i}}{\eta_i!}e^{-\lambda} \ . \tag{3.5}$$

Brownian energy process BEP($2k$)
The process with generator $\mathcal{L}_0^{BEP}$ has reversible measures given by product of `Gamma` *distributions with parameters $2k > 0$ and arbitrary $\theta > 0$, i.e.*

$$\mathbb{P}(dz) = \prod_{i=1}^{L} \frac{1}{(\theta)^{2k}\Gamma(2k)} z_i^{2k-1} e^{-z_i/\theta} dz_i \ . \tag{3.6}$$

3.2 Equilibrium product measure with reservoirs

We recall that, in the case of particle systems (see Sec.2), the reservoirs are modeled by birth-death processes with creation rate $b(n)$ and annihilation rate $d(n)$, n the number of particles at the boundary. Each reservoir has, thus, its own reversible invariant probability measure, $p(n)$, which satisfies the detailed balance condition $b(n)p(n) = d(n+1)p(n+1)$. This condition can be used to compute $p(n)$. The average value of the random number n (that we call density, irrespective to its value) is the quantity imposed by the reservoir to the system.
The effects of the reservoirs, under the equilibrium conditions, are described in the following proposition, which can easily be proved with an explicit computation.

Proposition 3.2. *For the processes with generator $\mathcal{L}$ defined in Sec. 2 we have:*

Inclusion walkers SIP($2k$)
The left reservoir is modeled by the birth and death process with rates

$$b(n) = \alpha(2k+n), \quad d(n) = \gamma n, \quad n \in \mathbb{N} \ . \tag{3.7}$$

The stationary state of this reservoir is given by a Negative Binomial measure with parameters $2k$ and $p = \frac{\alpha}{\gamma}$. The reservoir density is $\rho := \langle n \rangle = 2k\frac{p}{1-p} = 2k\frac{\alpha}{\gamma-\alpha}$. The boundary driven process with generator $\mathcal{L}^{SIP}$ defined in (2.4), with parameters α, γ and β, δ such that $\alpha\beta - \gamma\delta = 0$ (and thus $\rho_a = \rho_b$) admits the stationary product distribution:

$$\otimes_{i=1}^{L} \ \texttt{Negative-Binom}(2k, p) \qquad \textit{with} \qquad p := \frac{\alpha}{\gamma} = \frac{\delta}{\beta} \qquad \textit{for} \quad \alpha < \gamma \quad \textit{and} \quad \delta < \beta \tag{3.8}$$

Exclusion walkers SEP ($2j$)
The left reservoir is modeled by

$$b(n) = \alpha(2j-n), \quad d(n) = \gamma n, \quad n \in \{0, 1, \ldots, 2j\}. \tag{3.9}$$

The stationary state of this reservoir is given by a Binomial measure with parameters $2j$ and $p = \frac{\alpha}{\gamma+\alpha}$. The reservoir density is $\rho := \langle n \rangle = 2jp = 2j\frac{\alpha}{\gamma+\alpha}$. The boundary driven process with generator $\mathcal{L}^{SEP}$ defined in (2.5), with parameters α, γ and β, δ such that $\alpha\beta - \gamma\delta = 0$ (and thus $\rho_a = \rho_b$) admits the stationary product distribution:

$$\otimes_{i=1}^{L} \ \texttt{Binom}(2j, p) \qquad \textit{with} \qquad p := \frac{\alpha}{\gamma+\alpha} = \frac{\delta}{\beta+\delta} \ . \tag{3.10}$$

Independent random walkers IRW
The left reservoir has a constant birth rate

$$b(n) = \alpha, \quad d(n) = \gamma n, \quad n \in \mathbb{N} \ . \tag{3.11}$$

This reservoir imposes a Poisson measure with parameter $\lambda = \frac{\alpha}{\gamma}$. Therefore the density (i.e. mean number of particle) is $\rho := \langle n \rangle = \frac{\alpha}{\gamma}$. If $\frac{\alpha}{\gamma} = \frac{\delta}{\beta}$ the process with generator $\mathcal{L}^{IRW}$ defined in (2.6) admits the stationary product measure:

$$\otimes_{i=1}^{L} \ \texttt{Poisson}(\lambda) \qquad \textit{with} \qquad \lambda := \frac{\alpha}{\gamma} = \frac{\delta}{\beta} \ . \tag{3.12}$$

Brownian energy process BEP($2k$)
In this case the generator of the left reservoir is :

$$\mathcal{L}_a^{BEP} \quad = \quad T_a \left(2k\frac{\partial}{\partial z} + y\frac{\partial^2}{\partial z^2} \right) - \frac{1}{2} z \frac{\partial}{\partial z}, \qquad z \in \mathbb{R}^+, \tag{3.13}$$

The stationary measure of this reservoir is the Gamma distribution with parameters $2k$ and $\theta = 2T_a$. From the properties of the Gamma distribution one has $\langle z \rangle = 4kT$. If $T_a = T_b$ then the process with generator $\mathcal{L}^{BEP}$ defined in (2.7) admits the stationary product measure:

$$\otimes_{i=1}^{L} \; \texttt{Gamma}\,(2k, 2T) \qquad \textit{with} \qquad T := T_a = T_b \; . \tag{3.14}$$

4 Duality

When the reservoirs of our boundary driven processes work at different parameters value so that different densities or temperatures are imposed on the two sides, the stationary measure is in general unknown. Remarkable exceptions are the boundary driven SEP(1), with at most one particle per site, for which a matrix product solution is available [DEHP], and the case of IRW, where the product structure of the equilibrium invariant measure is preserved.
An alternative approach to characterize the stationary non-equilibrium state is provided by duality. In section 4.1 we describe duality for the processes previously defined. Dual processes have absorbing boundaries at two extra sites with suitable absorbing rates depending on the parameters reservoirs. In general the duality functions are related to moments of the stationary distribution. In section 4.2 we show several applications of duality and we obtain via duality the stationary non-equilibrium measure of independent random walkers.

4.1 Dual processes

Consider the extended chain $\{0, 1, ..., L, L+1\}$ obtained from the original one by adding the bounday sites $\{0, L+1\}$. Let $\eta = (\eta_1, ..., \eta_L)$ be the configuration in the original process, we denote by $\xi = (\xi_0, \xi_1, ..., \xi_L, \xi_{L+1}) \in \Omega^{dual}$ the configuration for the dual process, where the configuration space Ω_{Dual} will be specified later. We say that $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are dual with duality function $D(\eta, \xi)$ if

$$\mathbb{E}_\eta[D(\eta_t, \xi)] = \mathbb{E}_\xi[D(\eta, \xi_t)] \qquad \text{for any} \quad t \geq 0, \quad (\eta, \xi) \in \Omega \times \Omega_{Dual} \; , \tag{4.1}$$

where $\mathbb{E}_\eta$ denotes the expectation in the original process started from the configuration η, whereas $\mathbb{E}_\xi$ denotes the expectation in the dual process started from the configuration ξ.

Theorem 4.1. *For the processes defined in Sec. 2 we have the following duality results.*

Inclusion walkers SIP($2k$). *The process $(\eta_t)_{t\geq 0}$ defined by* (2.4) *is dual to the absorbing boundaries process $(\xi_t)_{t\geq 0}$ with configuration space $\Omega_{Dual} = \mathbb{N}_0^{L+2}$ with generator*

$$\begin{aligned} \mathcal{L}_{Dual}^{SIP} f(\xi) \quad &= \quad (\gamma - \alpha)\xi_1 \left[f(\xi^{1,0}) - f(\xi) \right] \\ &+ \quad \sum_{i=1}^{L-1} \xi_i (2k + \xi_{i+1}) \left[f(\xi^{i,i+1}) - f(\xi) \right] + \xi_{i+1}(2k + \xi_i) \left[f(\xi^{i+1,i}) - f(\xi) \right] \\ &+ \quad (\beta - \delta)\xi_L \left[f(\xi^{L,L+1}) - f(\xi) \right] \; , \end{aligned} \tag{4.2}$$

with duality function

$$D^{SIP}(\eta, \xi) = \left(\frac{\alpha}{\gamma - \alpha} \right)^{\xi_0} \prod_{i=1}^{L} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \left(\frac{\delta}{\beta - \delta} \right)^{\xi_{L+1}} \; . \tag{4.3}$$

Exclusion walkers SEP$(2j)$. *The process $(\eta_t)_{t\geq 0}$ defined by (2.5) is dual to the absorbing boundaries process $(\xi_t)_{t\geq 0}$ with configuration space $\Omega_{Dual} = \mathbb{N}_0 \times \{0, 1, \ldots, 2j\}^L \times \mathbb{N}_0$ with generator*

$$\begin{aligned} \mathcal{L}^{SEP}_{Dual} f(\xi) &= (\alpha+\gamma)\xi_1 \left[f(\xi^{1,0}) - f(\xi)\right] \\ &+ \sum_{i=1}^{L-1} \xi_i(2j-\xi_{i+1}) \left[f(\xi^{i,i+1}) - f(\xi)\right] + \xi_{i+1}(2j-\xi_i) \left[f(\xi^{i+1,i}) - f(\xi)\right] \\ &+ (\beta+\delta)\xi_L \left[f(\xi^{L,L+1}) - f(\xi)\right] , \end{aligned} \tag{4.4}$$

with duality function

$$D^{SEP}(\eta, \xi) = \left(\frac{\alpha}{\alpha+\gamma}\right)^{\xi_0} \prod_{i=1}^{L} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2j+1-\xi_i)}{\Gamma(2j+1)} \left(\frac{\delta}{\beta+\delta}\right)^{\xi_{L+1}} . \tag{4.5}$$

Independent random walkers IRW. *The process $(\eta_t)_{t\geq 0}$ defined by (2.6) is dual to the absorbing boundaries process $(\xi_t)_{t\geq 0}$ with configuration space $\Omega_{Dual} = \mathbb{N}_0^{L+2}$ with generator*

$$\begin{aligned} \mathcal{L}^{IRW}_{Dual} f(\xi) &= \gamma\xi_1 \left[f(\xi^{1,0}) - f(\xi)\right] \\ &+ \sum_{i=1}^{L-1} \xi_i \left[f(\xi^{i,i+1}) - f(\xi)\right] + \xi_{i+1} \left[f(\xi^{i+1,i}) - f(\xi)\right] \\ &+ \beta\xi_L \left[f(\xi^{L,L+1}) - f(\xi)\right] , \end{aligned} \tag{4.6}$$

with duality function

$$D^{ind}(\eta, \xi) = \left(\frac{\alpha}{\gamma}\right)^{\xi_0} \prod_{i=1}^{L} \frac{\eta_i!}{(\eta_i - \xi_i)!} \left(\frac{\delta}{\beta}\right)^{\xi_{L+1}} . \tag{4.7}$$

Brownian energy process BEP$(2k)$. *The process $(z_t)_{t\geq 0}$ defined by (2.7) is dual to the absorbing boundary process $(\xi_t)_{t\geq 0}$ with configuration space $\Omega_{Dual} = \mathbb{N}_0^{L+2}$ with generator*

$$\begin{aligned} \mathcal{L}^{BEP}_{Dual} f(\xi) &= \frac{\xi_1}{2} \left[f(\xi^{1,0}) - f(\xi)\right] \\ &+ \sum_{i=1}^{L-1} \xi_i(2k+\xi_{i+1}) \left[f(\xi^{i,i+1}) - f(\xi)\right] + \xi_{i+1}(2k+\xi_i) \left[f(\xi^{i+1,i}) - f(\xi)\right] \\ &+ \frac{\xi_L}{2} \left[f(\xi^{L,L+1}) - f(\xi)\right] , \end{aligned} \tag{4.8}$$

the duality function is

$$D^{BEP}(z, \xi) = (2T_a)^{\xi_0} \prod_{i=1}^{L} z_i^{\xi_i} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} (2T_b)^{\xi_{L+1}} . \tag{4.9}$$

Theorem 4.1 can be proven by explicit computations checking that the effect of the generator of a process on duality functions is the same as the effect of the generator of the dual process. Hereafter we only include the proof of the duality property for the inclusion process, being the proofs for the other processes quite similar.

Proof of Duality for the SIP$(2k)$. From [GKRV] we know that the free boundary inclusion process (i.e. the process generated by the operator $\mathcal{L}_0^{SIP}$ defined in (2.4)) is self-dual with duality function:

$$D_0^{SIP}(\eta, \xi) = \prod_{i=1}^{L} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} \tag{4.10}$$

this means that the action of $\mathcal{L}_0^{SIP}$ on $D_0^{SIP}(\cdot,\xi)$ and on $D_0^{SIP}(\eta,\cdot)$ is the same, i.e.

$$[\mathcal{L}_0^{SIP} D_0^{SIP}(\cdot,\xi)](\eta) = [\mathcal{L}_0^{SIP} D_0^{SIP}(\eta,\cdot)](\xi) \tag{4.11}$$

thus, since $\mathcal{L}_0^{SIP}$ does not act on the 0-th and $L+1$-th components of ξ, we have

$$[\mathcal{L}_0^{SIP} D^{SIP}(\cdot,\xi)](\eta) = [\mathcal{L}_0^{SIP} D^{SIP}(\eta,\cdot)](\xi) \tag{4.12}$$

It remains to verify that the actions of the operators $\mathcal{L}^{SIP}$ and $\mathcal{L}_{\mathrm{Dual}}^{SIP}$ at the boundaries are the same on the duality function. We verify this for the left boundary:

$$\begin{aligned}
[\mathcal{L}_a^{SIP} D^{SIP}(\cdot,\xi)](\eta) &= \alpha(2k+\eta_1)\left[D^{SIP}(\eta^{0,1},\xi) - D^{SIP}(\eta,\xi)\right] + \gamma\eta_1\left[D^{SIP}(\eta^{1,0},\xi) - D^{SIP}(\eta,\xi)\right] \\
&= D^{SIP}(\eta,\xi)\,\frac{(\eta_1-\xi_1)!}{\eta_1!}\cdot\Bigg\{\alpha(2k+\eta_1)\left[\frac{(\eta_1+1)!}{(\eta_1+1-\xi_1)!} - \frac{\eta_1!}{(\eta_1-\xi_1)!}\right] \\
&\qquad + \gamma\eta_1\left[\frac{(\eta_1-1)!}{(\eta_1-1-\xi_1)!} - \frac{\eta_1!}{(\eta_1-\xi_1)!}\right]\Bigg\} \\
&= D^{SIP}(\eta,\xi)\,\frac{\xi_1}{(\eta_1+1-\xi_1)}\cdot\{\alpha(2k+\eta_1) - \gamma(\eta_1+1-\xi_1)\} \\
&= D^{SIP}(\eta,\xi)\,\frac{\xi_1}{(\eta_1+1-\xi_1)}\cdot\{\alpha(2k+\xi_1-1) - (\gamma-\alpha)(\eta_1+1-\xi_1)\} \\
&= \alpha\,\frac{(2k+\xi_1-1)}{(\eta_1+1-\xi_1)}\,\xi_1\,D^{SIP}(\eta,\xi) - (\gamma-\alpha)\,\xi_1\,D^{SIP}(\eta,\xi) \\
&= (\gamma-\alpha)\xi_1\left[D^{SIP}(\eta,\xi^{1,0}) - D^{SIP}(\eta,\xi)\right] = [\mathcal{L}_{\mathrm{Dual},a}^{SIP} D^{SIP}(\eta,\cdot)](\xi)
\end{aligned} \tag{4.13}$$

We have used the notations $\mathcal{L}_{\mathrm{a}}^{SIP}$ and $\mathcal{L}_{\mathrm{Dual,a}}^{SIP}$ to denote the left boundary parts of the generators $\mathcal{L}^{SIP}$ and $\mathcal{L}_{\mathrm{Dual}}^{SIP}$ (i.e. the first line in (2.4), resp. (4.2)). By an analogous computation it is possible to verify that

$$[\mathcal{L}_b^{SIP} D^{SIP}(\cdot,\xi)](\eta) = [\mathcal{L}_{\mathrm{Dual},b}^{SIP} D^{SIP}(\eta,\cdot)](\xi) \tag{4.14}$$

where $\mathcal{L}_{\mathrm{b}}^{SIP}$ and $\mathcal{L}_{\mathrm{Dual,b}}^{SIP}$ are the right boundary parts of the two generators. This concludes the proof of the duality property. □

Remark 4.2. *At this point one may wonder whether there exists a diffusion process dual to the SEP. All the attempts that we have done in this direction seem to suggest that this is not the case. On the other hand, one may extend the definition of duality at the level of the generators, i.e. we say that the operator $\mathcal{L}$ is dual to the operator $\mathcal{L}_{Dual}$ with duality function $D(z,\eta)$ if*

$$[\mathcal{L}D(z,\cdot)](\eta) = [\mathcal{L}_{Dual}D(\cdot,\eta)](z)\ . \tag{4.15}$$

Notice that this definition does not require $\mathcal{L}$ and $\mathcal{L}_{Dual}$ to be Markov generators. Under this definition, it turns out that the SEP$(2j)$ free boundary operator $\mathcal{L}_0^{SEP}$ is "dual" to the differential operator defined in (2.16) *that has been obtained as a scaling limit of the SEP$(2j)$.*

4.2 Moments and duality.

In this section we provide some applications of duality. Since the dual process voids the chain, we show that the problem of computing stationary expectations for the original process is reduced to the computation of the absorption probabilities at the boundaries of the dual walkers. In partricular, we will see how the n-points correlations are related to the absorption probabilities at the extra sites 0 and $L+1$ of n dual walkers.

4.2.1 Stationary expectations and absorption probabilities

In the following Proposition we provide a relation connecting the expectation of the duality function and the absorption probabilities of the dual walkers.

Proposition 4.3. *Let $\langle\cdot\rangle_L$ denote expectation with respect to the stationary measure of the processes defined in Section 2. Let $(\xi(t))_{t\geq 0}$ denote the dual processes defined in Theorem 4.1. For a given $\xi \in \Omega_{Dual}$ let $|\xi| = \sum_{i=0}^{L+1} \xi_i$ and define $a_m(\xi)$ the absorption probabilities of the corresponding dual walkers initialized at ξ (i.e. ξ_i dual walkers start from site i), namely*

$$a_m(\xi) = \mathbb{P}(\{\xi_0(\infty) = m, \xi_{L+1}(\infty) = |\xi| - m\} \mid \{\xi_i(0) = \xi_i, \quad \forall i = 1, \dots, L\}) \,. \tag{4.16}$$

Then we have: in the case of the boundary driven processes $SIP(2k)$, $SEP(2j)$ and IRW

$$\langle D(\eta,\xi)\rangle_L = \sum_{m=0}^{|\xi|} (c\rho_a)^m \, (c\rho_b)^{|\xi|-m} \, a_m(\xi) \,, \tag{4.17}$$

where $c = \frac{1}{2k}$ for $SIP(2k)$ model, $c = \frac{1}{2j}$ for $SEP(2j)$ model, $c = 1$ for IRW model, and where the densities ρ_a and ρ_b are defined in Table 1; in the case of the boundary driven processes $BEP(2k)$

$$\langle D(z,\xi)\rangle_L = \sum_{m=0}^{|\xi|} (2T_a)^m \, (2T_b)^{|\xi|-m} \, a_m(\xi) \,. \tag{4.18}$$

Proof. We prove (4.17). Let μ_{L,ρ_a,ρ_b} be the stationary measure of the process η with boundary densities ρ_a and ρ_b. From the definition of duality in (4.1) and exploiting the fact that the dual walkers are absorbed at the boundaries, we have

$$\begin{aligned}
\langle D(\eta,\xi)\rangle_L &= \int D(\eta,\xi)\, \mu_{L,\rho_a,\rho_b}(d\eta) \\
&= \lim_{t\to\infty} \mathbb{E}_\eta \left[D(\eta_t,\xi)\right] \\
&= \lim_{t\to\infty} \mathbb{E}_\xi \left[D(\eta,\xi_t)\right] \\
&= \sum_{m=0}^{|\xi|} (c\rho_a)^m \, (c\rho_b)^{|\xi|-m} \, \mathbb{P}_\xi \left(\{\xi_0(\infty) = m, \xi_{L+1}(\infty) = |\xi| - m\}\right) ,
\end{aligned} \tag{4.19}$$

where $\mathbb{P}_\xi$ is the probability law of the dual process $(\xi(t))_{t\geq 0}$ started at ξ at time zero, and the last identity follows from the formulas of the duality functions (4.3), (4.5), (4.7) and the definitions of the densities given in Table 1. The proof of (4.18) is analogous. □

4.2.2 Averages in the stationary state

In this section we will see that all the boundary driven stochastic models considered so far have a linear density or temperature profile i.e. the expectations $\langle\eta_i\rangle$ or $\langle z_i\rangle$ with respect to the stationary measure is a linear function of i. This is an immediate consequence of duality since, in order to study the average at site i in the original process, it is enough to consider a single dual random walker started at i and it is an elementary fact that its absorption probabilities at the boundaries will be linear in i. Let us see.

For a system of size L, the expectations $\langle\eta_i\rangle_L$ and $\langle z_i\rangle_L$ can be written, up to a factor, as the expectations (with respect to the stationary measures of the processes $\eta(t)$ and $z(t)$) of the duality functions $D(\eta,\xi^i)$ computed in the configuration ξ^i with $\xi^i_j = \delta_{i,j}$. Furthermore, using Proposition

4.3, they can be explicitly found as functions of the dual absorption probabilities $p_i := a_1(\xi^i)$ and $a_0(\xi^i) = 1 - p_i$ ($a_m(\xi)$ as in Proposition 4.3). We have

$$\langle \eta_i \rangle_L = \frac{1}{c} \langle D(\eta, \xi^i) \rangle_L = \rho_a \, p_i + \rho_b \, (1 - p_i) \qquad i = 1, \ldots, L \tag{4.20}$$

for SIP, SEP and IRW, with c as in Proposition 4.3. Moreover, denoting by $\theta_a = 4kT_a$ and $\theta_b = 4kT_b$, we have

$$\langle z_i \rangle_L = 2k \langle D(z, \xi^i) \rangle_L = \theta_a \, p_i + \theta_b \, (1 - p_i) \qquad i = 1, \ldots, L \tag{4.21}$$

for the BEP. It remains to compute p_i.
Let X_t be the random walker moving on the chain $\{0, 1, \ldots, L+1\}$ as follows. In the bulk X_t jumps to one of the neighboring sites with rate $1/c$ (with c as in Proposition (4.3)), whereas it is absorbed by the left boundary (site 0) with rate u and by the right boundary (site $L+1$) with rate v. The values of c, u and v depend on the model, they are listed in Table (2).

System	SIP	SEP	IRW	BEP
u	$\gamma - \alpha$	$\gamma + \alpha$	γ	$1/2$
v	$\beta - \delta$	$\beta + \delta$	β	$1/2$
c	$1/2k$	$1/2j$	1	$1/2k$

Table 2: Dual processes jump rates.

The value p_i can then be interpreted as the probability for the walker X_t started at i to be absorbed by the left boundary, i.e. $p_i = \mathbb{P}(X_\infty = 0 \mid X_0 = i)$. They verify the following system of equations:

$$\begin{cases} p_0 = 1 \\ p_1 = \frac{1}{cu+1} \, p_2 + \frac{cu}{cu+1} \, p_0 \\ p_{i-1} - 2p_i + p_{i+1} = 0, \quad i = 2, \ldots, L-1 \\ p_L = \frac{cv}{cv+1} \, p_{L+1} + \frac{1}{cv+1} \, p_{L-1} \\ p_{L+1} = 0 \; . \end{cases} \tag{4.22}$$

Thus p_i is a linear function of i for $1 \le i \le L$ and the solution of (4.22) is given by:

$$p_i = \frac{L + \frac{1}{cv} - i}{L + \frac{1}{cu} + \frac{1}{cv} - 1} \qquad \text{for} \quad i = 1, \ldots, L \qquad \text{and} \quad p_0 = 1, \;\; p_{L+1} = 0 \; . \tag{4.23}$$

Hence, from (4.20), for SIP, SEP, and IRW we get

$$\langle \eta_i \rangle_L = \frac{\rho_a \left(L + \frac{1}{cv} - i \right) + \left(i + \frac{1}{cu} - 1 \right) \rho_b}{L + \frac{1}{cu} + \frac{1}{cv} - 1} \qquad i = 1, \ldots, L \tag{4.24}$$

with u, v as in Table (2), and ρ_a, ρ_b as in Table (1).

Remark 4.4. *Under a suitable rescaling of the constants tuning the annihilation rates at the boundaries (see Remark 2.3), the solutions of the exclusion and of the inclusion walkers scale to those of the independent walkers.*

Finally, from (4.21) and (4.23), for the BEP we get

$$\langle z_i \rangle = \frac{\theta_a (L + 4k - i) + \theta_b (i - 1 + 4k)}{L + 8k - 1} \qquad i = 1, \ldots, L \; . \tag{4.25}$$

and, by a similar computation, we find

$$\langle z_i \rangle = \frac{T_a(2L-3-2i)+T_b(2i-1)}{2(L-2)} \qquad i=1,\ldots,L\ . \tag{4.26}$$

for the KMP model.

4.2.3 Stationary product measure for the boundary driven independent walkers

In the following proposition the stationary measure for the boundary driven IRW is obtained as an application of the duality property.

Proposition 4.5. *The stationary measure of the process with generator $\mathcal{L}^{IRW}$ defined in 2.6 is the product measure with marginals at each site $i=1,\ldots,L$ given by Poisson distribution with parameter*

$$\lambda_i = \frac{\rho_a\left(L+\frac{1}{\beta}-i\right)+\rho_b\left(i-1+\frac{1}{\gamma}\right)}{L+\frac{1}{\beta}+\frac{1}{\gamma}-1}\ . \tag{4.27}$$

Proof. Since for a random variable X with Poisson distribution of parameter λ the n^{th} factorial moment is given by $\mathbb{E}(X(X-1)\ldots(X-n+1))=\lambda^n$, to prove the proposition is enough to check the identity

$$\langle \prod_{i=1}^{L}\frac{\eta_i!}{(\eta_i-\xi_i)!}\rangle_L = \prod_{i=1}^{L}\lambda_i^{\xi_i}\ . \tag{4.28}$$

To this aim consider a dual walker that starts his walk from site $i\in\{1,\ldots,L\}$. The probability p_i of its ultimate absorption at site 0 is given by

$$p_i = \frac{L+\frac{1}{\beta}-i}{L+\frac{1}{\beta}+\frac{1}{\gamma}-1} \tag{4.29}$$

(see (4.23) and Table (2)). Using formula (4.17) and observing that the absorption probabilities of a total of $\sum_{i=1}^{L}\xi_i$ dual walkers, with ξ_i of them initialized at site i, completely factorize because the walkers are independent, one has

$$\begin{aligned}\langle \prod_{i=1}^{L}\frac{\eta_i!}{(\eta_i-\xi_i)!}\rangle_L &= \prod_{i=1}^{L}\sum_{m_i=0}^{\xi_i}\rho_a^{m_i}\rho_b^{\xi_i-m_i}\binom{\xi_i}{m_i}p_i^{m_i}(1-p_i)^{\xi_i-m_i}\\ &= \prod_{i=1}^{L}\left(\rho_a p_i+\rho_b(1-p_i)\right)^{\xi_i}\ .\end{aligned}$$

Inserting (4.29) in the above formula and remembering the definition of the λ_i, equation (4.28) is verified and the proof of the proposition is completed. □

4.2.4 Duality moment functions

It turns out from the previous section that the expectations of the duality functions $D(\eta_t,\xi)$ with respect to the probability law of the original process η_t, i.e. the *"duality moment functions"*

$$G(\eta,\xi,t) := \mathbb{E}_\eta\left[D(\eta_t,\xi)\right] \tag{4.30}$$

are usually some kind of moments of the original process η_t labelled by the discrete parameter $\xi\in\Omega_{dual}$. In the case of SEP, SIP and IRW, the function $G(\eta,\xi,t)$ is, up to a multiplicative constant depending on ξ, the ξ-th factorial moment at time t when the initial value is η. In the case of BEP, the function $G(z,\xi,t) := \mathbb{E}_z[D(z_t,\xi)]$ is the standard ξ-th moment. Under suitable conditions, the set of moments, obtained on varying the parameter ξ, completely characterizes the law of the original process. From duality we find that the equations for the functions $G(\eta,\xi,t)$ are closed and quite simple to write.

Proposition 4.6. *Let η_t and ξ_t be two dual Markov processes with duality function $D(\eta,\xi)$ and let $\mathcal{L}$ and $\mathcal{L}_{Dual}$ be their generators, then the duality moment function $G(\eta,\xi,t)$ defined in* (4.30) *satisfies the following equation:*

$$\frac{d}{dt}G(\eta,\xi,t) = [\mathcal{L}_{Dual}G(\eta,\cdot,t)](\xi)\ . \tag{4.31}$$

Proof. For any function $f = f(\eta)$ we have

$$\frac{d}{dt}\mathbb{E}_\eta[f(\eta_t)] = \mathbb{E}_\eta[\mathcal{L}f(\eta_t)]\ . \tag{4.32}$$

Given $\xi \in \Omega_{dual}$, applying (4.32) to $f(\eta) = D(\eta,\xi)$ and using duality, namely $[\mathcal{L}D(\cdot,\xi)](\eta) = [\mathcal{L}_{\rm Dual}D(\eta,\cdot)](\xi)$ one has

$$\frac{d}{dt}\mathbb{E}_\eta[D(\eta_t,\xi)] = \mathbb{E}_\eta[[\mathcal{L}_{\rm Dual}D(\eta_t,\cdot)](\xi)] = [\mathcal{L}_{\rm Dual}\,\mathbb{E}_\eta[D(\eta_t,\cdot)]](\xi). \tag{4.33}$$

Equation (4.31) follows from the definition of the function G (cfr. (4.30)). □

Corollary 4.7. *Let $\langle\cdot\rangle$ denote expectation in the stationary state and define the* "stationary duality moment functions"

$$G(\xi) := \langle D(\eta,\xi)\rangle\ . \tag{4.34}$$

It immediately follows from Proposition 4.6 that $G(\xi)$ satisfies the equation

$$(L_{Dual}G)(\xi) = 0\ . \tag{4.35}$$

We will see an application of the function G in section 5.2.

5 Instantaneous thermalization and KMP model

In this Section we define the boundary driven process with instantaneous thermalization. An *instantaneous thermalization* process gives rise, for each couple of nearest neighboring sites, to an instantaneous redistribution of the total energy (or of the total number of particles). The class of instantaneous thermalization processes we consider in this paper is obtained from the non-equilibrium processes defined so far after performing a suitable "instantaneous thermalization limit": for each bond, the total energy E (or the total number of particles) of that bond is redistributed according to the stationary measure of the original process at equilibrium on that bond, conditioned to the conservation of E.

5.1 Thermalized models

To start with we recall a well known instantaneous thermalization model, the KMP model (see [KMP]). The KMP model is defined by considering on each bond a uniform redistribution of energy. At the boundaries the energy is fixed by a reservoir which imposes a Boltzmann-Gibbs exponential energy distribution with different temperatures T_a and T_b. The generator of the process is

$$\begin{aligned}
\mathcal{L}^{KMP} f(z) &= \int_0^\infty dz_1' \, \frac{e^{-z_1'/T_a}}{T_a} \left(f(z_1', z_2, \ldots, z_L) - f(z)\right) \\
&+ \sum_{i=1}^{L-1} \int_0^1 dx \left(f(z_1, \ldots, x(z_i + z_{i+1}), (1-x)(z_i + z_{i+1}), \ldots, z_L) - f(z)\right) \\
&+ \int_0^\infty dz_L' \, \frac{e^{-z_L'/T_b}}{T_b} \left(f(z_1, \ldots, z_{L-1}, z_L') - f(z)\right)
\end{aligned} \tag{5.1}$$

for any $f : \mathbb{R}_+^L \to \mathbb{R}$.
At the end of this section we will see that the KMP model can be obtained as the instantaneous thermalization limit of the BEP$(2k)$ model in the particular case $k = 1/2$.
From [KMP] we know that the KMP is dual to a suitable discrete Markov process. The dual process $\xi(t) = (\xi_0(t), \xi_1(t), \ldots, \xi_L(t), \xi_{L+1}(t)) \in \mathbb{N}_0^{L+2}$ describes the motion of particles in a one dimensional $L+2$ -sites chain. The boundary sites ξ_0 and ξ_{L+1} are absorbing. In the bulk, for each couple of neighboring sites $(i, i+1)$ there is an instantaneous uniform redistribution of the total number of particles $\xi_i + \xi_{i+1}$. The redistribution takes place whenever an exponentially distributed clock rings. The clocks (one for each couple $(i, i+1)$) are mutually independent. The generator of this process is defined on functions $f : \mathbb{N}_0^{L+2} \to \mathbb{R}$ by

$$\begin{aligned}
\mathcal{L}_{\text{Dual}}^{KMP} f(\xi) &= \left[f(\xi_0 + \xi_1, 0, \xi_2, \ldots, \xi_{L+1}) - f(\xi)\right] \\
&+ \sum_{i=1}^{L-1} \sum_{r=0}^{\xi_i + \xi_{i+1}} \left[f(\xi_0, \ldots, \xi_{i-1}, r, \xi_i + \xi_{i+1} - r, \ldots, \xi_{L+1}) - f(\xi)\right] \\
&+ \left[f(\xi_0, \ldots, 0, \xi_L + \xi_{L+1}) - f(\xi)\right] \,,
\end{aligned} \tag{5.2}$$

and the duality function is $D^{\text{KMP}}(z, \xi) = T_a^{\xi_0} \prod_{i=1}^N \frac{z_i^{\xi_i}}{\xi_i!} \, T_b^{\xi_{L+1}}$.

We will see that, for each of the instantaneous thermalization processes that we are going to introduce there is a dual process. The dual processes are instantaneous thermalization processes themselves. They have absorbing boundaries and can be naturally derived by a thermalization limit from the dual processes of the original ones (see Section 4).

Thermalized Inclusion walkers Th-SIP$(2k)$. The instantaneous thermalization limit of the Inclusion process is obtained as follows. Imagine on each bond $(i, i+1)$ to run the SIP$(2k)$ dynamics for an infinite amount of time. Then the total number of particles on the bond will be redistributed according to the stationary measure on that bond, conditioned to conservation of the total number of particles of the bond. We consider two independent random variables η_i and η_{i+1} distributed according to the stationary measure of the SIP$(2k)$ at the equilibrium. Thus η_i and η_{i+1} are two Negative Binomial random variables of parameters $2k$ and p. Hence $\eta_i + \eta_{i+1}$ is again a Negative Binomial r.v. with parameters $4k$ and p and then the distribution of one of them, given that the sum is fixed to $\eta_i + \eta_{i+1} = E$, has a Negative Hypergeometric probability density of parameters $(E, 4k-1, 2k)$, i.e.

$$\nu_{2k}^{SIP}(r \,|\, E) := \mathbb{P}(\eta_1 = r \,|\, \eta_i + \eta_{i+1} = E) = \frac{\binom{2k+r-1}{r} \cdot \binom{2k+E-r-1}{E-r}}{\binom{4k+E-1}{E}} \,. \tag{5.3}$$

On the other hand, the stationary distribution of the left Inclusion reservoir is the Negative Binomial with parameters $2k$ and $\frac{\alpha}{\gamma}$ (resp. $\frac{\delta}{\beta}$ for the right reservoir). Then the generator of the instantaneous thermalization limit of the Inclusion process with reservoirs can be defined as follows

$$\begin{aligned}
\mathcal{L}_{th}^{SIP} f(\eta) &= \sum_{r=0}^{\infty} [f(r,\eta_2,\dots,\eta_L) - f(\eta)] \binom{2k+r-1}{r} \left(\frac{\alpha}{\gamma}\right)^r \left(\frac{\gamma-\alpha}{\gamma}\right)^{2k} \\
&+ \sum_{i=1}^{L-1} \sum_{r=0}^{\eta_i+\eta_{i+1}} [f(\eta_1,\dots,\eta_{i-1},r,\eta_i+\eta_{i+1}-r,\eta_{i+2},\dots,\eta_L) - f(\eta)] \ \nu_{2k}^{SIP}(r \,|\, \eta_i+\eta_{i+1}) \\
&+ \sum_{r=0}^{\infty} [f(\eta_1,\dots,\eta_{L-1},r) - f(\eta)] \binom{2k+r-1}{r} \left(\frac{\delta}{\beta}\right)^r \left(\frac{\beta-\delta}{\beta}\right)^{2k} . \qquad (5.4)
\end{aligned}$$

It is easy to check that the thermalized inclusion process is dual, with duality function (4.3), to the process that behaves in the bulk as the thermalized SIP$(2k)$ itself, and which has absorbing boundaries at two extra sites with absorbing rate 1. In other words the dual process is generated by:

$$\begin{aligned}
\mathcal{L}_{th,\mathrm{Dual}}^{SIP} f(\xi) &= [f(\xi_0+\xi_1,0,\xi_2,\dots,\xi_{L+1}) - f(\xi)] \\
&+ \sum_{i=1}^{L-1} \sum_{r=0}^{\xi_i+\xi_{i+1}} [f(\xi_0,\dots,\xi_{i-1},r,\xi_i+\xi_{i+1}-r,\xi_{i+2},\dots,\xi_{L+1}) - f(\xi)] \ \nu_{2k}^{SIP}(r \,|\, \xi_i+\xi_{i+1}) \\
&+ [f(\xi_0,\dots,\xi_{L-1},0,\xi_L+\xi_{L+1}) - f(\xi)] \ , \qquad (5.5)
\end{aligned}$$

where $\xi = (\xi_0,\xi_1,\dots,\xi_L,\xi_{L+1})$.

Thermalized Exclusion walkers Th-SEP$(2k)$. If we take two independent random variables η_i and η_{i+1} with Binomial distribution of parameters $2j$ and p, then $\eta_i+\eta_{i+1}$ is again a Binomial r.v. with parameters $4j$ and p; then the distribution of one of them, given the sum fixed to $\eta_i+\eta_{i+1}=E$, has an Hypergeometric distribution with parameters $(E,4j,2j)$, i.e. a probability mass function

$$\nu_{2j}^{SEP}(r \,|\, E) := \mathbb{P}(\eta_1 = r \,|\, \eta_1+\eta_2 = E) = \frac{\binom{2j}{r}\cdot\binom{2j}{E-r}}{\binom{4j}{E}} \mathbf{1}_{r\le 2j} \ . \qquad (5.6)$$

The stationary distribution of the left Exclusion reservoir is the Binomial with parameters $2j$ and $\frac{\alpha}{\gamma+\alpha}$ (resp. $\frac{\delta}{\beta+\delta}$ for the right reservoir). Then we define the generator of the instantaneous thermalization limit of the Exclusion process with reservoirs as follows

$$\begin{aligned}
\mathcal{L}_{th}^{SEP} f(\eta) &= \sum_{r=0}^{2j} [f(r,\eta_2,\dots,\eta_L) - f(\eta)] \binom{2j}{r} \left(\frac{\alpha}{\gamma}\right)^r \left(\frac{\gamma}{\gamma+\alpha}\right)^{2j} \\
&+ \sum_{i=1}^{L-1} \sum_{r=0}^{\eta_i+\eta_{i+1}} [f(\eta_1,\dots,\eta_{i-1},r,\eta_i+\eta_{i+1}-r,\eta_{i+2},\dots,\eta_L) - f(\eta)] \ \nu_{2j}^{SEP}(r \,|\, \eta_i+\eta_{i+1}) \\
&+ \sum_{r=0}^{2j} [f(\eta_1,\dots,\eta_{L-1},r) - f(\eta)] \binom{2j}{r} \left(\frac{\delta}{\beta}\right)^r \left(\frac{\beta}{\beta+\delta}\right)^{2j} . \qquad (5.7)
\end{aligned}$$

The thermalized exclusion process is dual, with duality function (4.5), to the process that behaves in the bulk as the process itself, and which has absorbing boundaries at two extra sites with absorbing rate 1:

$$\begin{aligned}
\mathcal{L}_{th,\mathrm{Dual}}^{SEP} f(\xi) &= [f(\xi_0+\xi_1,0,\xi_2,\dots,\xi_{L+1}) - f(\xi)] \\
&+ \sum_{i=1}^{L-1} \sum_{r=0}^{\xi_i+\xi_{i+1}} [f(\xi_1,\dots,\xi_{i-1},r,\xi_i+\xi_{i+1}-r,\xi_{i+2},\dots,\xi_L) - f(\xi)] \ \nu_{2j}^{SEP}(r \,|\, \xi_i+\xi_{i+1}) \\
&+ [f(\xi_0,\dots,\xi_{L-1},0,\xi_L+\xi_{L+1}) - f(\xi)] \ , \qquad (5.8)
\end{aligned}$$

where $\xi = (\xi_0, \xi_1, \dots, \xi_L, \xi_{L+1})$.

Thermalized Indepent walkers Th-IRW. Let η_i and η_{i+1} be two independent random variables with Poisson distribution of parameter λ, then $\eta_i + \eta_{i+1}$ is again a Poisson r.v. with parameter 2λ then the distribution of one of them, given the sum fixed to $\eta_i + \eta_{i+1} = E$, has a Binomial density of parameters $(E, 1/2)$:

$$\nu^{IRW}(r \,|\, E) := \mathbb{P}(\eta_1 = r \,|\, \eta_1 + \eta_2 = E) = \binom{E}{r} \frac{1}{2^E} \,. \tag{5.9}$$

Moreover the stationary distribution of the left IRW reservoir is the Poisson with parameter $\frac{\alpha}{\gamma}$ (resp. $\frac{\delta}{\beta}$ for the right reservoir). Then the generator of the instantaneous thermalization limit of the independent walkers process with reservoirs is given by

$$\begin{aligned}
\mathcal{L}_{th}^{IRW} f(\eta) &= \sum_{r=0}^{\infty} [f(r, \eta_2, \dots, \eta_L) - f(\eta)] \left(\frac{\alpha}{\gamma}\right)^r \frac{e^{-\alpha/\gamma}}{r!} \\
+ \sum_{i=1}^{L-1} \sum_{r=0}^{\eta_i + \eta_{i+1}} & [f(\eta_1, \dots, \eta_{i-1}, r, \eta_i + \eta_{i+1} - r, \eta_{i+2}, \dots, \eta_L) - f(\eta)] \; \nu^{IRW}(r \,|\, \eta_i + \eta_{i+1}) \\
&+ \sum_{r=0}^{\infty} [f(\eta_1, \dots, \eta_{L-1}, r) - f(\eta)] \left(\frac{\delta}{\beta}\right)^r \frac{e^{-\delta/\beta}}{r!} \,.
\end{aligned} \tag{5.10}$$

The thermalized independent walkers process is dual, with duality function (4.7), to the process that behaves in the bulk as the process itself, and which has absorbing boundaries at two extra sites:

$$\begin{aligned}
\mathcal{L}_{th,\mathrm{Dual}}^{IRW} f(\xi) &= [f(\xi_0 + \xi_1, 0, \xi_2, \dots, \xi_{L+1}) - f(\xi)] \\
+ \sum_{i=1}^{L-1} \sum_{r=0}^{\eta_i + \eta_{i+1}} & [f(\xi_1, \dots, \xi_{i-1}, r, \xi_i + \xi_{i+1} - r, \xi_{i+2}, \dots, \xi_L) - f(\xi)] \; \nu^{IRW}(r \,|\, \xi_i + \xi_{i+1}) \\
&+ [f(\xi_0, \dots, \xi_{L-1}, 0, \xi_L + \xi_{L+1}) - f(\xi)] \,,
\end{aligned} \tag{5.11}$$

where $\xi = (\xi_0, \xi_1, \dots, \xi_L, \xi_{L+1})$.

Thermalized Brownian energy process Th-BEP($2k$). We define the instantaneous thermalization limit of the Brownian Energy process as follows. On each bond we run the BEP($2k$) for an infinite time. Then the energies on the bond will be redistributed according to the stationary measure on that bond, conditioned to the conservation of the total energy of the bond. If we take two independent random variables z_i and z_{i+1} with Gamma distribution of parameters $2k$ and θ, then the distribution of one of them, given the sum fixed to $z_i + z_{i+1} = E$, has density

$$p(z_i | z_i + z_{i+1} = E) = \frac{z_i^{2k-1} (E - z_i)^{2k-1}}{\int_0^E z_i^{2k-1} (E - z_i)^{2k-1} dz_i} \,. \tag{5.12}$$

Equivalently, the variable $x = z_i/E$ has a Beta($2k, 2k$) distribution. Denoting by $\nu_{2k}^{BEP}(x)$ the density of such a random variable, we can define the generator of the instantaneous thermalization limit of the Brownian Energy process with reservoirs as follows

$$\begin{aligned}
\mathcal{L}_{th}^{BEP} f(z) &= \int_0^\infty dz_1' \, \frac{1}{(T_a)^{2k} \Gamma(2k)} (z_1')^{2k-1} e^{-z_1'/T_a} \left(f(z_1', z_2, \dots, z_L) - f(z)\right) \qquad (5.13) \\
&+ \sum_{i=1}^{L-1} \int_0^1 [f(z_1, \dots, x(z_i + z_{i+1}), (1-x)(z_i + z_{i+1}), \dots, z_L) - f(z)] \; \nu_{2k}^{BEP}(x) dx \\
&+ \int_0^\infty dz_L' \, \frac{1}{(T_b)^{2k} \Gamma(2k)} (z_L')^{2k-1} e^{-z_L'/T_b} \left(f(z_1, \dots, z_{L-1}, z_L') - f(z)\right) \,.
\end{aligned}$$

Remark 5.1. *For $k = 1/2$ this reproduces the uniform redistribution rule of the KMP model on each bond of the bulk. The same is true for the reservoirs since the stationary distribution of the Brownian Energy reservoir is Gamma with parameters $2k$ and θ. If one takes $k = 1/2$, then one obtains an Exponential distribution with parameter θ.*

The thermalized Brownian Energy process is dual, with duality function (4.9) to the process that behaves in the bulk as the thermalized inclusion process, and which has absorbing boundaries at two extra sites with absorbing rate 1. In other words the dual process is generated by:

$$\begin{aligned}\mathcal{L}^{BEP}_{th,\mathrm{Dual}} f(\xi) &= [f(\xi_0+\xi_1, 0, \xi_2, \ldots, \xi_{L+1}) - f(\xi)] \\ &+ \sum_{i=1}^{L-1} \sum_{r=0}^{\xi_i+\xi_{i+1}} [f(\xi_0, \ldots, \xi_{i-1}, r, \xi_i+\xi_{i+1}-r, \xi_{i+2}, \ldots, \xi_{L+1}) - f(\xi)]\ \nu^{SIP}_{2k}(r \,|\, \xi_i+\xi_{i+1}) \\ &+ [f(\xi_0, \ldots, \xi_{L-1}, 0, \xi_L+\xi_{L+1}) - f(\xi)]\ ,\end{aligned} \tag{5.14}$$

where $\xi = (\xi_0, \xi_1, \ldots, \xi_L, \xi_{L+1})$.

Remark 5.2. *It is immediately seen that for $k = 1/2$, this gives the KMP-dual process defined in* (5.2).

5.2 Stationary measures for $L = 1$.

In this section we compute the moments of the istantaneous thermalization processes defined so far, by using the result obtained in Corollary 4.7. When $L = 1$ there is no bulk contribution in the generator, since we have a site interacting with two sources.
The $L = 1$ case is trivial for our original processes (SEP, SIP, IRW and BEP), because for these models the contributions of the baths are additive. Then the system is indeed equivalent to the system of one site interacting with one bath whose parameters are recombinations of the parameters of the two original baths. The stationary measure is, then, the stationary measure of this total bath.
The interest of the $L = 1$ case for the thermalized processes lies in the fact that for these models the baths contribution are no longer additive. Thus, even in this basic case the computation of the stationary measure is worth to be investigated. A result in this direction has been obtained in [BDGJL] (see also the remark at the end of this section).

Thermalized Inclusion walkers Th-SIP($2k$). From the previous section we know that the thermalized inclusion process η_t is dual to the process defined in (5.5), with duality function (4.3), then the duality moment function is

$$G^{SIP}_{th}(\xi) = \langle D^{SIP}(\eta, \xi) \rangle = \left(\frac{\alpha}{\gamma-\alpha}\right)^{\xi_0} \frac{\Gamma(2k)}{\Gamma(2k+\xi_1)} \left(\frac{\delta}{\beta-\delta}\right)^{\xi_2} M(\xi_1)\ , \tag{5.15}$$

where $M(\xi)$ is the ξ-th factorial moment with respect to the stationary measure of η_1:

$$M(\xi_1) = \langle \frac{\eta_1!}{(\eta_1-\xi_1)!} \rangle\ . \tag{5.16}$$

In order to find $M(\xi)$, from Corollary 4.7 we impose

$$\begin{aligned} 0 &= \mathcal{L}^{SIP}_{th,\mathrm{Dual}} G(\xi) = G(\xi_0+\xi_1, 0, \xi_2) - 2G(\xi_0, \xi_1, \xi_2) + G(\xi_0, 0, \xi_1+\xi_2) \\ &= \left(\frac{\alpha}{\gamma-\alpha}\right)^{\xi_0} \left(\frac{\delta}{\beta-\delta}\right)^{\xi_2} \cdot \left\{ \left(\frac{\alpha}{\gamma-\alpha}\right)^{\xi_1} - 2M(\xi_1)\frac{\Gamma(2k)}{\Gamma(2k+\xi_1)} + \left(\frac{\delta}{\beta-\delta}\right)^{\xi_1} \right\}\ .\end{aligned}$$

This yields

$$M(\xi_1) = \frac{\Gamma(2k+\xi_1)}{2\Gamma(2k)} \left[\left(\frac{\alpha}{\gamma-\alpha} \right)^{\xi_1} + \left(\frac{\delta}{\beta-\delta} \right)^{\xi_1} \right] . \tag{5.17}$$

Thermalized Exclusion walkers Th-SEP$(2j)$. The thermalized inclusion process η_t is dual to the process in (5.8), with duality function (4.5), then the stationary duality moment function is

$$G^{SEP}_{th}(\xi) = \langle D^{SEP}(\eta,\xi) \rangle = \left(\frac{\alpha}{\gamma+\alpha} \right)^{\xi_0} \frac{\Gamma(2j+1-\xi_1)}{\Gamma(2j+1)} \left(\frac{\delta}{\beta+\delta} \right)^{\xi_2} M(\xi_1) , \tag{5.18}$$

where $M(\xi)$ is the ξ-th factorial moment with respect to the stationary measure ν of η_1:

$$M(\xi_1) = \langle \frac{\eta_1!}{(\eta_1-\xi_1)!} \rangle . \tag{5.19}$$

From Corollary 4.7, we have

$$\begin{aligned} 0 &= \mathcal{L}^{SEP}_{th,\text{Dual}} G(\xi) = G(\xi_0+\xi_1,0,\xi_2) - 2G(\xi_0,\xi_1,\xi_2) + G(\xi_0,0,\xi_1+\xi_2) \\ &= \left(\frac{\alpha}{\gamma+\alpha} \right)^{\xi_0} \left(\frac{\delta}{\beta+\delta} \right)^{\xi_2} \cdot \left\{ \left(\frac{\alpha}{\gamma+\alpha} \right)^{\xi_1} - 2M(\xi_1) \frac{\Gamma(2j+1-\xi_1)}{\Gamma(2j+1)} + \left(\frac{\delta}{\beta+\delta} \right)^{\xi_1} \right\} . \end{aligned}$$

This yields

$$M(\xi_1) = \frac{\Gamma(2j+1)}{2\Gamma(2j+1-\xi_1)} \cdot \left[\left(\frac{\alpha}{\gamma+\alpha} \right)^{\xi_1} + \left(\frac{\delta}{\beta+\delta} \right)^{\xi_1} \right] . \tag{5.20}$$

Thermalized independent random walkers Th-IRW. The thermalized IRW process η_t is dual to the process defined in (5.11), with duality function (4.7), then the stationary duality moment function is

$$G^{IRW}_{th}(\xi) = \langle D^{IRW}(\eta,\xi) \rangle = \left(\frac{\alpha}{\gamma} \right)^{\xi_0} \left(\frac{\delta}{\beta} \right)^{\xi_2} M(\xi_1) , \tag{5.21}$$

where $M(\xi_1)$ is the ξ-th factorial moment with respect to the stationary measure η_1 as above. To find $M(\xi_1)$ we impose

$$\begin{aligned} 0 &= \mathcal{L}^{IRW}_{th,\text{Dual}} G(\xi) = G(\xi_0+\xi_1,0,\xi_2) - 2G(\xi_0,\xi_1,\xi_2) + G(\xi_0,0,\xi_1+\xi_2) \\ &= \left(\frac{\alpha}{\gamma} \right)^{\xi_0} \left(\frac{\delta}{\beta} \right)^{\xi_2} \cdot \left\{ \left(\frac{\alpha}{\gamma} \right)^{\xi_1} - 2M(\xi_1) + \left(\frac{\delta}{\beta} \right)^{\xi_1} \right\} . \end{aligned}$$

This gives

$$M(\xi_1) = \frac{1}{2} \left[\left(\frac{\alpha}{\gamma} \right)^{\xi_1} + \left(\frac{\delta}{\beta} \right)^{\xi_1} \right] . \tag{5.22}$$

Thermalized brownian energy process Th-BEP$(2k)$. The thermalized brownian energy process z_t is dual to the process defined in (5.14), with duality function (4.9), then the stationary duality moment function is

$$G^{BEP}_{th}(\xi) = \langle D^{BEP}(\eta,\xi) \rangle = (2T_a)^{\xi_0} \frac{\Gamma(2k)}{\Gamma(2k+\xi_1)} (2T_b)^{\xi_2} M(\xi_1) \tag{5.23}$$

where $M(\xi_1)$ is now the ξ_1^{th} moment with respect to the stationary measure of z_1:

$$M(\xi_1) = \langle z_1^{\xi_1} \rangle \ . \tag{5.24}$$

In order to find $M(\xi_1)$, from Corollary 4.7 we impose

$$\begin{aligned} 0 &= \mathcal{L}^{BEP}_{th,\text{Dual}} G(\xi) = G(\xi_0 + \xi_1, 0, \xi_2) - 2G(\xi_0, \xi_1, \xi_2) + G(\xi_0, 0, \xi_1 + \xi_2) \\ &= (2T_a)^{\xi_0} \ (2T_b)^{\xi_2} \cdot \left\{ (2T_a)^{\xi_1} - 2M(\xi_1) \frac{\Gamma(2k)}{\Gamma(2k+\xi_1)} + (2T_b)^{\xi_1} \right\} \end{aligned}$$

This yields

$$M(\xi_1) = \frac{\Gamma(2k+\xi_1)}{2\Gamma(2k)} \cdot \left[(2T_a)^{\xi_1} + (2T_b)^{\xi_1} \right] \tag{5.25}$$

Remark 5.3. *For $k = 1/2$ the $M(\xi_1)$ above becomes*

$$M(\xi_1) = \frac{\xi_1!}{2} \cdot \left[(2T_a)^{\xi_1} + (2T_b)^{\xi_1} \right] \ . \tag{5.26}$$

The knowledge of all the moments fully describes the stationary measure of the KMP process with 1 particle. A similar result was obtained in [BDGJL]. In that paper an explicit form of the stationary measure for 1 particle connected to two reservoirs is given, however for a process which is slightly different that the original KMP process. The difference lies at the boundaries thermalization mechanism: in the KMP model the first and last sites are directly thermalized by the reservoirs, in [BDGJL] the first and last sites share uniformly their energies with thermalized reservoirs.

6 Correlations in the stationary state

For some of our processes, such as for the SEP(1) model [Spo], the BEP(1/2) model [GKR], the KMP model [BDGJL], the covariances have been proven to be bilinear. For the boundary driven SEP(1), from the exact solution of the microscopic stationary state (see e.g. [DE, DEHP, DLS1]), we know even more. Indeed for this process all the correlations $\langle \eta_{i_1} \dots \eta_{i_n} \rangle$ in the stationary state are multilinear in the variables $i_1, \dots, i_n$, and can be explicitely computed through a recursive argument on n and L by a matrix method.
The algebraic similarity of the generators for our whole class of models, that includes the SEP(1) , leads us to expect multilinear correlation functions. This turns out to be false in general, as we will see in this section. For instance bilinearity of the covariances holds only for a certain choice of the boundary parameters for the SEP($2j$) and for the SIP($2k$) and only in the case $k = 1/4$ for the BEP. From Proposition 4.3, multilinearity of the correlation functions is in turn implied by multilinearity of the absorption probabilities of the dual walkers.

In this section we compute the two points correlations w.r. to the stationary measure, for a suitable choice of the parameters. We do this by direct computation, i.e. for the particle models we require that the generator of the process vanishes on the functions $f(\eta) = \eta_i \, \eta_\ell$. This yields a linear system in the variables $X_{i,\ell} := \langle \eta_i \eta_\ell \rangle$, $i \le \ell$. Analogous computations, with η_i replaced by z_i, are performed for the BEP model.

6.1 Covariances

The correlations in the stationary state, i.e. the expectations $X_{i,\ell} = \langle \eta_i \eta_\ell \rangle$ with $1 \le i \le \ell \le L$, satisfy a system of $L(L+1)/2$ equations. The equations are quite complicated (we include them in the Appendix, see section (8)) and then hard to solve directly. What we found is that, for a generic choice of the boundary parameters, for none of our processes there exists a bilinear function satisfying them. In other words, the ansatz

$$X_{i,\ell} = Ai\ell + Bi + C\ell + D \qquad \text{for} \quad i < \ell \qquad\qquad \text{and} \qquad X_{i,i} = Ei^2 + Fi + G \tag{6.1}$$

does not produce a solution for the systems in (8.1) and (8.1) (since the number of independent equations that the coefficients in (6.1) should satisfy is larger than 7). But there exist some conditions on $\alpha, \beta, \gamma, \delta$ producing an effective simplification of the systems (8.1). Under this conditions the correlations for the SIP$(2k)$ and for the SEP$(2j)$ are bilinear and then explicitly computable through the ansatz (6.1). In what follows we provide such explicit bilinear forms. In order to verify their validity one can simply put the generic forms (6.1) in the systems (8.1), find the equations that must be satisfied by the 7 coefficients and solve them.
Finally, at the end of the paragraph, we will see by duality that one needs to fix $k = 1/4$ in order to have bilinear correlations for the BEP.

We denote by $\langle \eta_i \eta_\ell \rangle_c$ the covariances (truncated correlations) in the stationary state of the particle models, namely $\langle \eta_i \eta_\ell \rangle_c := \langle \eta_i \eta_\ell \rangle - \langle \eta_i \rangle \langle \eta_\ell \rangle$. Replacing η_i with z_i one defines the covariances of the BEP model.

Inclusion walkers SIP$(2k)$. If the parameters satisfy the condition

$$\gamma = 2k + \alpha \qquad \text{and} \qquad \beta = 2k + \delta \,, \tag{6.2}$$

i.e.

$$\rho_a = 2k \, \frac{\alpha}{\gamma - \alpha} = \alpha \qquad \text{and} \qquad \rho_b = 2k \, \frac{\delta}{\delta - \beta} = \delta \,, \tag{6.3}$$

one has

$$\langle \eta_i \, \eta_\ell \rangle_c \quad = \quad \frac{i(L+1-\ell)}{(L+1)^2(2k(L+1)+1)} \, (\rho_a - \rho_b)^2 \qquad \text{for} \qquad i < \ell \,, \tag{6.4}$$

whereas $\langle \eta_i^2 \rangle$ is a quadratic function of i. Notice that, under this same choice of parameters, the expression for the averages is simplified to

$$\langle \eta_i \rangle = \rho_a + (\rho_b - \rho_a) \frac{i}{L+1} \,. \tag{6.5}$$

Exclusion walkers SEP$(2j)$. Under the choice of the parameters

$$\gamma = 2j - \alpha \qquad \text{and} \qquad \beta = 2j - \delta \,, \tag{6.6}$$

i.e.

$$\rho_a = 2j \, \frac{\alpha}{\alpha + \gamma} = \alpha \qquad \text{and} \qquad \rho_b = 2j \, \frac{\delta}{\beta + \delta} = \delta \,, \tag{6.7}$$

the two points correlations are bilinear and they are given by

$$\langle \eta_i \, \eta_\ell \rangle_c \quad = \quad - \frac{i \, (L+1-\ell)}{(L+1)^2(2j(L+1)-1)} \, (\rho_a - \rho_b)^2 \qquad \text{for} \qquad i < \ell \,, \tag{6.8}$$

the variances are quadratic and the average profile becomes

$$\langle \eta_i \rangle = \rho_a + (\rho_b - \rho_a) \frac{i}{L+1} \,. \tag{6.9}$$

Brownian energy process BEP(1/2). In [GKR] it was studied the BEP model for $k = 1/4$ and it was found that the two points correlations are bilinear. For $i < \ell$ they are given by:

$$\langle z_i \, z_\ell \rangle_c \quad = \quad \frac{2i(L+1-\ell)}{(L+3)(L+1)^2} \, (\theta_b - \theta_a)^2 \; . \tag{6.10}$$

In this case one has the neat linear profile

$$\langle z_i \rangle = \theta_a + (\theta_b - \theta_a) \frac{i}{L+1} \; . \tag{6.11}$$

The result in (6.10) can be obtained from (6.4) and duality. Indeed, comparing (4.2) and (4.8), one notices that the dual processes of BEP($2k$) and SIP($2k$) with $\gamma - \alpha = 2k$ and $\beta - \delta = 2k$ do coincide if and only if $k = 1/4$. Under this choice, when the dual process is initialized from the configuration $\bar{\xi}$ having one particle at site i and one particle at site ℓ, equation (4.3) becomes

$$D^{SIP}(\eta, \bar{\xi}) = (2\alpha)^{\xi_0} \;\; 4\eta_i \eta_\ell \;\; (2\delta)^{\xi_{L+1}} \tag{6.12}$$

and equation (4.9) becomes

$$D^{BEP}(z, \bar{\xi}) = (2T_a)^{\xi_0} \; 4 z_i z_\ell \; (2T_b)^{\xi_{L+1}} \; . \tag{6.13}$$

Therefore, with this choice of parameters and initial conditions, the duality functions are the same if one identifies $T_a = \alpha = \rho_a$ and $T_b = \delta = \rho_b$ and the result (6.10) immediately follows from (6.4).

We can summarize the situation as follows. The covariances are bilinear at least in the following cases:

(a) SEP(1), ($j = 1/2$) and generic $\alpha, \beta, \gamma, \delta$.

(b) SEP($2j$) for $j \in \{1, 3/2, 2, 5/2, ...\}$ and $\gamma + \alpha = 2j$ and $\beta + \delta = 2j$.

(c) SIP($2k$) for $k > 0$ and $\gamma - \alpha = 2k$ and $\beta - \delta = 2k$.

(d) BEP($\frac{1}{2}$), ($k = 1/4$) and generic T_a, T_b.

We remark that the conditions (b),(c),(d) are those giving the neat average profile of equations (6.5), (6.9) and (6.11), i.e. those yielding exactly the densities ρ_a and ρ_b (resp. the temperatures T_a and T_b) in the proximity of the reservoirs (i.e. for $i = 0$ and $i = L + 1$).

The following further properties of the covariances are observed by solving the equations in the Appendix (8)) on *Mathematica*. As the parameters are varied, the covariances are:

(i) proportional to $(\rho_b - \rho_a)^2$ or $(T_b - T_a)^2$.

(ii) positive for the inclusion walkers and for the Brownian energy process, negative for the exclusion walkers: this is related to the attractive (bosonic) interaction of the first two system, compared to the repulsive (fermionic) interaction of exclusion. For the proof of this property see [GRV].

6.2 Results for the n-points correlations

The multilinearity of the correlations seems to be, thus, prerogative of some special cases. One may wonder about the multilinearity for the 3-points correlations, in the same range of parameters leading to bilinearity for the 2-points correlations (i.e. in the cases (b),(c),(d) above). The explicit solution of the n-points correlations problem becomes harder and harder as n increases and even the case $n = 3$ is quite difficult to solve exactly.

In this paragraph we provide the results of some numerical computations. We solved numerically the master equation for the invariant distribution of SEP$(2j)$ in the cases $L = 6$ and $j = 1/2, 1, 3/2, 2$ and computed the correlations $\langle \eta_i \eta_j \rangle_c$ and $\langle \eta_i \eta_j \eta_\ell \rangle_c$. If $\langle \eta_1 \eta_i \rangle_c$ were multilinear, the differences $d_i = \langle \eta_1 \eta_{i+1} \rangle_c - \langle \eta_1 \eta_i \rangle_c$, $i = 2, 3, 4, 5$ would be constant. The simulations seem to confirm the bilinearity of the covariances in the cases (a) and (b) above, and the loss of bilinearity in the other cases. In Fig.1 (left panel) the values of d_i are reported for the case $\alpha = 1$, $\gamma = 1$, $\beta = 1/2$, $\delta = 3/2$: they are clearly constant for $j = 1/2$ (case (a) above) and for $j = 1$ (case (b) above) but not for $j = 3/2$ and $j = 2$.

Concerning the 3-points correlations, the simulations show that the multilinearity is lost even in the cases where it holds for $n = 2$ (i.e. in the case (b)), while it is conserved for the SEP(1) with at most one particle per site. Figure 1 gives evidence for this phenomenon by showing that $e_i = \langle \eta_1 \eta_2 \eta_{i+1} \rangle_c - \langle \eta_1 \eta_2 \eta_i \rangle_c$, $i = 3, 4, 5$ are constant only for $j = 1/2$ (case (a) above).
The deviation from multilinearity is in any case very small and, very likely, it is decreasing as L increases.

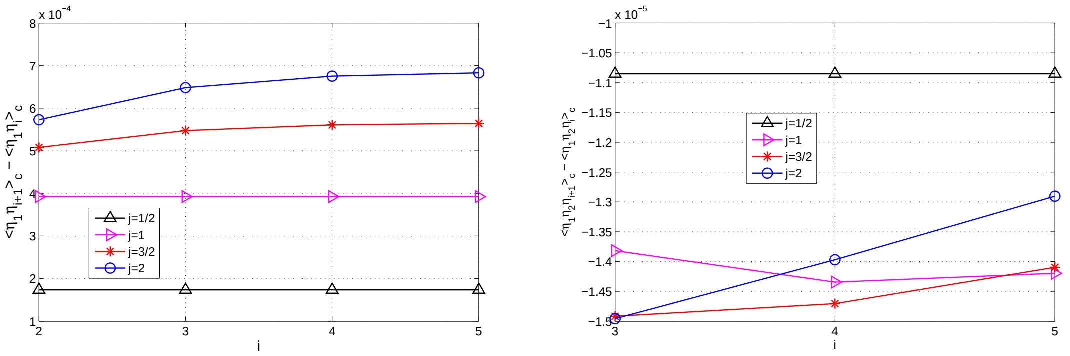

Figure 1: Test for the multilinearity of the connected correlations $\langle \eta_1 \eta_i \rangle_c$, $\langle \eta_1 \eta_2 \eta_i \rangle_c$ for SEP$(2j)$, $j = 1/2\,(\triangle)$, $j = 1\,(\triangleright)$, $j = 3/2\,(*)$, $j = 2\,(\circ)$ with $\alpha = 1$, $\gamma = 1$, $\beta = 1/2$, $\delta = 3/2$ and $L = 6$. In the left panel $d_i = \langle \eta_1 \eta_{i+1} \rangle_c - \langle \eta_1 \eta_i \rangle_c$, $i = 2, 3, 4, 5$; in the right panel $e_i = \langle \eta_1 \eta_2 \eta_{i+1} \rangle_c - \langle \eta_1 \eta_2 \eta_i \rangle_c$, $i = 3, 4, 5$. Non constant d_i or e_i imply violation of the multilinearity.

7 Hydrodynamic limit

The aim of this section is to show that the large scale properties of the models studied so far can be obtained by a suitable adaptation of the macroscopic fluctuation theory of [BDGJL, BDGJL1, BDGJL2, BDGJL3]. In particular we verify that the macroscopic limit of the exact solutions for the covariances found in Section 6 does match the prediction of the macroscopic fluctuation theory (see [DLS, DG] for the exclusion process with at most one particle per site).

7.1 Macroscopic fluctuation theory and density large deviations functional

We briefly review the approach of the macroscopic fluctuation theory. Let us consider a one dimensional diffusive systems of linear size L in contact with two reservoirs at densities ρ_a and ρ_b. The macroscopic fluctuation theory describes the behavior of the system in the hydrodynamic limit in terms of the two quantities $D(\rho)$ and $\sigma(\rho)$, called *diffusivity* and *mobility*, defined by

$$D(\rho) := \lim_{\delta\rho\to 0} \lim_{L\to\infty} \frac{L}{\delta\rho} \cdot \frac{\langle Q_{i,i+1}(t)\rangle_{L,\rho,\rho+\delta\rho}}{t} , \tag{7.1}$$

$$\sigma(\rho) := \lim_{L\to\infty} \frac{\langle Q^2_{i,i+1}(t)\rangle_{L,\rho,\rho}}{t} , \tag{7.2}$$

where

$$Q_{i,i+1}(t) = \int_0^t j_{i,i+1}(t')dt' . \tag{7.3}$$

In the above equation $Q_{i,i+1}(t)$ is the total flow through the bond $i, i+1$ in the time interval $[0,t]$, while $j_{i,i+1}(t')$ is the instantaneous flow at time t'. The bracket $\langle\cdot\rangle_{L,\rho_a,\rho_b}$ denotes the expectation with respect to the stationary state for the system of size L whose density on the left (resp. right) boundary is ρ_a (resp. ρ_b).
From the macroscopic fluctuation theory [BDGJL2], we know that the probability of observing a time dependent density and current profiles $\rho(x,\tau)$ and $j(x,\tau)$ in a macroscopic time interval $[\tau_1,\tau_2]$, under the diffusive scaling $x = i/L$ and $\tau = t/L^2$, is $\sim exp(-L\mathcal{A})$, where $\mathcal{A}$ is the action functional given by:

$$\mathcal{A}(\{\rho(x,s), j(x,s)\}; \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} ds \int_0^1 dx \frac{\left[j(x,s) + D(\rho(x,s))\dfrac{\partial\rho(x,s)}{\partial x}\right]^2}{2\sigma(\rho(x,s))} . \tag{7.4}$$

Then the probability of observing a density profile $\rho(x)$ in the stationary state is $\mathbb{P}(\rho(x)) \sim e^{-L\mathcal{F}(\rho(x))}$ where $\mathcal{F}$ is the large deviation functional:

$$\mathcal{F}(\rho(x)) = \min_{\{\rho(x,s),j(x,s)\}} \mathcal{A}(\{\rho(x,s), j(x,s)\}; -\infty, \tau) \tag{7.5}$$

with the minimum in (7.5) taken over all the trajectories conditioned to the extreme values $\rho(x,-\infty) = \rho^*(x)$, $\rho^*(x)$ the typical profile, and $\rho(x,\tau) = \rho(x)$. Density and current must also satisfy the continuity equation

$$\partial\rho/\partial\tau = -\,\partial j/\partial x. \tag{7.6}$$

The density correlation functions in the stationary state can be obtained from the large deviation functional $\mathcal{F}$ through the derivatives of its Legendre transform $\mathcal{G}$ (see [D]):

$$\mathcal{G}(\{\alpha(x)\}) = \sup_{\{\rho(x)\}} \left\{ \int_0^1 \alpha(x)\rho(x)\, dx - \mathcal{F}(\{\rho(x)\}) \right\} . \tag{7.7}$$

Then, for large L we have

$$\langle\rho(x)\rangle = \left.\frac{\partial\mathcal{G}}{\partial\alpha(x)}\right|_{\alpha(x)=0} \tag{7.8}$$

$$\begin{aligned} \langle\rho(x)\rho(y)\rangle_c &= \frac{1}{L} \left.\frac{\partial^2\mathcal{G}}{\partial\alpha(x)\,\partial\alpha(y)}\right|_{\alpha(x)=0} \\ \vdots \quad & \quad \vdots \\ \langle\rho(x_1)\rho(x_2)\,\ldots\,\rho(x_k)\rangle_c &= \frac{1}{L^{k-1}} \left.\frac{\partial^k\mathcal{G}}{\partial\alpha(x_1)\,\ldots\,\partial\alpha(x_k)}\right|_{\alpha(x)=0} . \end{aligned} \tag{7.9}$$

7.2 From SEP(1) to models with constant diffusivity and quadratic mobility

In this section we use a scaling argument to deduce the density large deviations functional of a model with constant diffusivity and quadratic mobility from that of the SEP(1) (cfr. also [BDGJL2]). We start by recalling that for the SEP(1) one has

$$D(\rho) = 1, \quad \sigma(\rho) = 2\rho(1-\rho) \,, \tag{7.10}$$

and therefore

$$\mathcal{A}_{SEP(1)}(\{\rho(x,s), j(x,s)\}; \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} ds \int_0^1 dx \frac{\left[j(x,s) + \frac{\partial \rho(x,s)}{\partial x}\right]^2}{4\rho(x,s)(1-\rho(x,s))} \tag{7.11}$$

then, from (7.5), one finds that the density large deviation functional is (see [DLS], [BDGJL3])

$$\mathcal{F}_{SEP(1)}(\{\rho(x)\}) = \sup_{F(x)} \int_0^1 dx \left[\rho(x) \log \frac{\rho(x)}{F(x)} + (1-\rho(x)) \log\left(\frac{1-\rho(x)}{1-F(x)}\right) + \log \frac{F'(x)}{\rho_a - \rho_b}\right] \tag{7.12}$$

where the supremum is taken over the monotone functions with boundary values $F(0) = \rho_a$, $F(1) = \rho_b$. The supremum is attained for $F = F_\rho$, monotone solution of the following differential problem:

$$\rho(x) = F + \frac{F(1-F)F''}{(F')^2} \qquad \text{with} \qquad F(0) = \rho_a \quad \text{and} \quad F(1) = \rho_b. \tag{7.13}$$

The connected correlation functions can be obtained by computing the derivatives of the functional $\mathcal{G}_{SEP(1)}$ as in (7.7). One finds that the lowest order correlations are, for large L,

$$\begin{aligned}
\langle \rho(x) \rangle^{SEP(1)} &= \rho_a(1-x) + \rho_b x \\
\langle \rho(x)\rho(y) \rangle_c^{SEP(1)} &= -\frac{(\rho_a-\rho_b)^2}{L} x(1-y) \\
\langle \rho(x)\rho(y)\rho(z) \rangle_c^{SEP(1)} &= -2\frac{(\rho_a-\rho_b)^3}{L^2} x(1-2y)(1-z),
\end{aligned} \tag{7.14}$$

for $0 < x < y < z$.

Let us now consider the generalization of (7.10) obtained by assuming that the diffusivity is constant and the mobility is a quadratic function parametrized as

$$D(\rho) = C, \quad \sigma(\rho) = 2A\rho(B-\rho), \tag{7.15}$$

where A, B and C are given numbers. The action functional of this system

$$\mathcal{A}(\{\rho(x,s), j(x,s)\}; \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} ds \int_0^1 dx \frac{\left[j(x,s) + C\frac{\partial \rho(x,s)}{\partial x}\right]^2}{4A\rho(x,s)(B-\rho(x,s))} \tag{7.16}$$

is related to $\mathcal{A}_{SEP(1)}$ through the following change of variables (cfr. [DG])

$$\mathcal{A}(\{\rho(x,\tau), j(x,\tau)\}; \tau_1, \tau_2) = \frac{C}{A} \mathcal{A}_{SEP(1)}(\{\tilde\rho(x,s), \tilde j(x,s)\}; C\tau_1, C\tau_2) \,, \tag{7.17}$$

with

$$\tilde\rho(x,s) := \frac{1}{B}\, \rho(x, C^{-1}s) \qquad \text{and} \qquad \tilde j(x,s) := \frac{1}{BC}\, j(x, C^{-1}s) \tag{7.18}$$

The scaling (7.17) has been chosen among all the possible scalings connecting $\mathcal{A}$ and $\mathcal{A}_{SEP(1)}$ as it is the only one preserving the conservation law (7.6) between $\tilde{\rho}(x,s)$ and $\tilde{j}(x,s)$.
Then, by (7.17) and (7.5) it follows that the large deviation functional for the system characterized by (7.15) is given by

$$\mathcal{F}(\{\rho(x)\}) = \frac{C}{A}\,\mathcal{F}_{SEP(1)}\left(\{B^{-1}\rho(x)\}\right)\ , \tag{7.19}$$

and thus

$$\mathcal{F}(\{\rho(x)\}) = \frac{C}{AB}\sup_{\tilde{F}(x)}\int_0^1 dx\left[\rho(x)\log\frac{\rho(x)}{\tilde{F}(x)} + (B-\rho(x))\log\left(\frac{B-\rho(x)}{B-\tilde{F}(x)}\right) + B\log\frac{\tilde{F}'(x)}{\tilde{\rho}_a-\tilde{\rho}_b}\right]\ , \tag{7.20}$$

with

$$\tilde{\rho}_a = B\rho_a, \qquad \tilde{\rho}_b = B\rho_b \qquad \text{and} \qquad \tilde{F}(x) = BF(x)\ , \tag{7.21}$$

where $F(x)$ is the monotone function satisfying (7.13). Equivalently $\tilde{F}$ is the monotone solution of the differential problem

$$\rho(x) = \tilde{F} + \frac{\tilde{F}(B-\tilde{F})\tilde{F}''}{(\tilde{F}')^2} \qquad \text{with} \qquad \tilde{F}(0)=\tilde{\rho}_a \qquad \text{and} \qquad \tilde{F}(1)=\tilde{\rho}_b. \tag{7.22}$$

Using (7.7) and (7.19) we find

$$\begin{aligned}
\mathcal{G}\left(\{\alpha(x)\}\right) &= \sup_{\{\tilde{\rho}(x)\}}\left\{\int_0^1 \alpha(x)\tilde{\rho}(x)\,dx - \mathcal{F}(\{\tilde{\rho}(x)\})\right\} \\
&= \sup_{\{\tilde{\rho}(x)\}}\left\{\int_0^1 \alpha(x)\tilde{\rho}(x)\,dx - \frac{C}{A}\,\mathcal{F}_{SEP(1)}(\{B^{-1}\tilde{\rho}(x)\})\right\} \\
&= \frac{C}{A}\sup_{\{\rho(x)\}}\left\{\int_0^1 \frac{AB}{C}\,\alpha(x)\rho(x)\,dx - \mathcal{F}_{SEP(1)}(\{\rho(x)\})\right\} \\
&= \frac{C}{A}\,\mathcal{G}_{SEP(1)}\left(\{ABC^{-1}\alpha(x)\}\right)
\end{aligned} \tag{7.23}$$

and, from (7.8) and (7.14), we have

$$\langle\rho(x)\rangle = \left.\frac{\partial\mathcal{G}}{\partial\alpha(x)}\right|_{\alpha(x)=0} = B\,\langle\rho(x)\rangle^{SEP(1)} = \tilde{\rho}_a(1-x)+\tilde{\rho}_b x \tag{7.24}$$

$$\langle\rho(x)\rho(y)\rangle_c = \frac{1}{L}\left.\frac{\partial^2\mathcal{G}}{\partial\alpha(x)\,\partial\alpha(y)}\right|_{\alpha(x)=0} = \frac{AB^2}{C}\,\langle\rho(x)\rho(y)\rangle_c^{SEP(1)} = -\left(\frac{A}{C}\right)\frac{(\tilde{\rho}_a-\tilde{\rho}_b)^2}{L}x(1-y)$$

$$\begin{aligned}
\langle\rho(x)\rho(y)\rho(z)\rangle_c &= \frac{1}{L^2}\left.\frac{\partial^3\mathcal{G}}{\partial\alpha(x)\,\partial\alpha(y)\partial\alpha(z)}\right|_{\alpha(x)=0} = \frac{A^2B^3}{C^2}\,\langle\rho(x)\rho(y)\rho(z)\rangle_c^{SEP(1)} \\
&= -2\left(\frac{A}{C}\right)^2\frac{(\tilde{\rho}_a-\tilde{\rho}_b)^3}{L^2}x(1-2y)(1-z)
\end{aligned}$$

and, more generally, one gets a factor $B^n(A/C)^{n-1}$ for the n-point connected correlation function.

7.3 Macroscopic behavior of the correlations

With suitable choices of the parameters A, B, C we can generate the large scale limits of models that we have considered in the previous sections.

Inclusion walkers SIP($2k$). For interacting particle systems, the flux across bond $(i, i+1)$ in a time interval $[0, t]$ is given by the number of particles which jump from i to $i+1$ minus the number of particles which jump from $i+1$ to i. i.e.

$$\begin{aligned} Q_{i,i+1}(t) &= \int_0^t dt' \left[\eta_{i+1}(t')(2k+\eta_i(t')) - \eta_i(t')(2k+\eta_{i+1}(t'))\right] \\ &= 2k \int_0^t dt' \left[\eta_{i+1}(t') - \eta_i(t')\right] \ . \end{aligned} \tag{7.25}$$

As a consequence, the expectation of the flow $Q_{i,i+1}(t)$ in the stationary state with boundary densities ρ_a, ρ_b is given by

$$\langle Q_{i,i+1}(t)\rangle_{L,\rho_a,\rho_b} = 2k \cdot t \cdot \langle \eta_{i+1} - \eta_i \rangle_{L,\rho_a,\rho_b} \ . \tag{7.26}$$

It follows, from (7.1) and (4.24), that $D(\rho) = 2k$.
From Section 3.2 we know that the SIP($2k$) equilibrium stationary measure at density ρ is the product of `NegativeBinomial` $(2k, \rho/(\rho+2k))$ with a variance $Var(\eta_i) = \frac{\rho(\rho+2k)}{2k}$. Using (7.25)

$$\langle Q^2_{i,i+1}(t)\rangle_{L,\rho,\rho} = (2k)^2 \int_0^t dt' \int_0^t dt'' \langle [\eta_{i+1}(t') - \eta_i(t')]\, [\eta_{i+1}(t'') - \eta_i(t'')]\rangle_{L,\rho,\rho} \ . \tag{7.27}$$

Now we have

$$\begin{aligned} & \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \int_0^t dt'' \langle (\eta_{i+1}(t') - \eta_i(t'))(\eta_{i+1}(t'') - \eta_i(t''))\rangle_{L,\rho,\rho} \\ = \ & \lim_{t\to\infty} \frac{2}{t} \int_0^t dt' \int_{t'}^t dt'' \langle (\eta_{i+1}(t') - \eta_i(t'))(\eta_{i+1}(t'') - \eta_i(t''))\rangle_{L,\rho,\rho} \\ = \ & 2 \int_0^\infty dt \, \langle (\eta_{i+1}(0) - \eta_i(0))(\eta_{i+1}(t) - \eta_i(t))\rangle_{L,\rho,\rho} \\ = \ & 2 \int_0^\infty dt \cdot \sum_\eta \left\{ (\eta_{i+1} - \eta_i)\, \mathbb{E}_\eta \left[\eta_{i+1}(t) - \eta_i(t)\right] \ \mu_{L,\rho,\rho}(\eta) \right\} \end{aligned} \tag{7.28}$$

where, in the last display, $\mu_{L,\rho,\rho}$ denotes the stationary equilibrium measure. By duality,

$$\begin{aligned} \mathbb{E}_\eta \left[\eta_{i+1}(t) - \eta_i(t)\right] &= 2k \left\{ \mathbb{E}_\eta \left[D_0^{SIP}(\eta(t), \xi^{i+1})\right] - \mathbb{E}_\eta \left[D_0^{SIP}(\eta(t), \xi^i)\right] \right\} \\ &= 2k \left\{ \mathbb{E}_{\xi^{i+1}} \left[D_0^{SIP}(\eta, \xi(t))\right] - \mathbb{E}_{\xi^i} \left[D_0^{SIP}(\eta, \xi(t))\right] \right\} \end{aligned} \tag{7.29}$$

where ξ^i is the L-dimensional configuration $(\xi^i_1, \ldots, \xi^i_L)$ with $\xi^i_j = \delta_{i,j}$ and D_0^{SIP} is the duality function defined in (4.10). Let $p_t(i,j)$ be the transition probability from the site i to the site j in the time interval $[0, t]$ of a random walker on the set $\{1, \ldots, L\}$ moving at rate $2k$, then

$$\mathbb{E}_{\xi^i} \left[D_0^{SIP}(\eta, \xi(t))\right] = \frac{1}{2k} \sum_j \eta_j \cdot p_t(i,j) \ . \tag{7.30}$$

As a consequence (7.28) is equal to

$$
\begin{aligned}
&2\sum_{j=1}^{L}\langle(\eta_{i+1}-\eta_i)\eta_j\rangle_{L,\rho,\rho}\cdot\int_0^\infty dt\ (p_t(i+1,j)-p_t(i,j))\\
&=4\,\langle(\eta_{i+1}-\eta_i)\eta_{i+1}\rangle_{L,\rho,\rho}\cdot\int_0^\infty dt\ (p_t(i+1,i+1)-p_t(i,i+1))\\
&=4\,Var(\eta_i)\cdot\int_0^\infty dt\ (p_t(0,0)-p_t(0,1)) \qquad (7.31)
\end{aligned}
$$

where the two identities above follow from the product character of the equilibrium measure, and from the fact that $p_t(i,j)$ depends only on the distance $|i-j|$. Now the random walk p_t is moving at rate $2k$, then, from the master equation we have

$$
2\,(p_t(0,0)-p_t(0,1)) = -\,(p_t(0,1)+p_t(0,-1)-2p_t(0,0)) = -\frac{1}{2k}\cdot\frac{d}{dt}p_t(0,0)\ . \qquad (7.32)
$$

Then (7.31) is given by

$$
-\,2Var(\eta_i)\cdot\int_0^\infty \frac{1}{2k}\frac{d}{dt}\,p_t(0,0)\cdot dt = 2\,\frac{1}{(2k)^2}\cdot\rho(\rho+2k)\cdot(1-p_\infty(0,0))\ . \qquad (7.33)
$$

Since $p_\infty(0,0)$ vanishes as $L\to\infty$, we finally obtain, using (7.2) $\sigma(\rho)=2\rho(\rho+2k)$. Summarizing, for the inclusion process SIP$(2k)$, we have

$$
D(\rho)=2k,\quad \sigma(\rho)=2\rho(\rho+2k)\ , \qquad (7.34)
$$

which implies $A=-1$, $B=-2k$ and $C=2k$. This choice produces (see (7.24)) the following correlation functions:

$$
\begin{aligned}
\langle\rho(x)\rangle &= \rho_a(1-x)+\rho_b x\\
\langle\rho(x)\rho(y)\rangle_c &= \frac{1}{2k}\frac{(\rho_a-\rho_b)^2}{L}x(1-y)\\
\langle\rho(x)\rho(y)\rho(z)\rangle_c &= -\left(\frac{1}{2k}\right)^2\frac{2(\rho_a-\rho_b)^3}{L^2}x(1-2y)(1-z). \qquad (7.35)
\end{aligned}
$$

where ρ_a and ρ_b are the SIP$(2k)$ boundary densities ($\rho_a = 2k\,\alpha/(\gamma-\alpha)$ and $\rho_b = 2k\,\delta/(\beta-\delta)$ in terms of our boundary parameters). Notice that the covariances in (7.35) do indeed agree with the macroscopic limit of the microscopic covariances that have been found in Section 6.1 (see (6.4)) for a particular choice of the parameters. Similarly, one gets for the density large deviation functional:

$$
\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx\left[\rho(x)\log\frac{\rho(x)}{F(x)}+(2k+\rho(x))\log\left(\frac{2k+\rho(x)}{2k+F(x)}\right)+2k\log\frac{F'(x)}{\rho_a-\rho_b}\right]\ , \qquad (7.36)
$$

where $F=F_\rho$ is the monotone solution of

$$
\rho(x) = F+\frac{F(2k+F)F''}{(F')^2} \qquad \text{with} \qquad F(0)=\rho_a \qquad \text{and} \qquad F(1)=\rho_b. \qquad (7.37)
$$

Exclusion walkers SEP$(2j)$. The flux is now given by

$$
\begin{aligned}
Q_{i,i+1}(t) &= \int_0^t dt'\,[\eta_{i+1}(t')(2j-\eta_i(t'))-\eta_i(t')(2j-\eta_{i+1}(t'))] \qquad (7.38)\\
&= 2j\int_0^t dt'\,[\eta_{i+1}(t')-\eta_i(t')]\ . \qquad (7.39)
\end{aligned}
$$

As a consequence, the expectation of $Q_{i,i+1}(t)$ with respect to the steady state measure reads

$$\langle Q_{i,i+1}(t)\rangle_{L,\rho_a,\rho_b} = 2j \cdot t \cdot \langle \eta_{i+1} - \eta_i \rangle_{L,\rho_a,\rho_b} \, . \tag{7.40}$$

Thus, from (7.1) and (4.24), we get $D(\rho) = 2j$. From Section 3.2 we know that the SEP$(2j)$ stationary measure at density ρ is the product of `Binomial` $(2j, \rho/2j)$ with a variance $Var(\eta_i) = \frac{\rho(2j-\rho)}{2j}$. Using a similar computation as for the inclusion walkers then, one can compute also the mobility, obtaining:

$$D(\rho) = 2j, \quad \sigma(\rho) = 2\rho(2j - \rho). \tag{7.41}$$

Therefore we have $A = 1$, $B = 2j$, $C = 2j$ and, from (7.24), we have the following correlation functions:

$$\begin{aligned}
\langle \rho(x) \rangle &= \rho_a(1-x) + \rho_b x \\
\langle \rho(x)\rho(y) \rangle_c &= -\frac{1}{2j}\frac{(\rho_a - \rho_b)^2}{L} x(1-y) \\
\langle \rho(x)\rho(y)\rho(z) \rangle_c &= -\left(\frac{1}{2j}\right)^2 \frac{2(\rho_a-\rho_b)^3}{L^2} x(1-2y)(1-z)
\end{aligned} \tag{7.42}$$

where ρ_a and ρ_b are the SEP$(2j)$ boundary densities ($\rho_a = 2j\,\alpha/(\alpha+\gamma)$ and $\rho_b = 2j\,\delta/(\beta+\delta)$ in terms of our boundary parameters). The second line in (7.42) does agree with the microscopic SEP-covariances that have been found in Section 6.1 (see (6.8)) for a particular choice of the parameters. Moreover the density large deviation functional is given by

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left[\rho(x)\log\frac{\rho(x)}{F(x)} + (2j - \rho(x))\log\left(\frac{2j-\rho(x)}{2j-F(x)}\right) + 2j\log\frac{F'(x)}{\rho_a - \rho_b}\right] \tag{7.43}$$

where $F = F_\rho$ is the monotone function satisfying

$$\rho(x) = F + \frac{F(2j-F)F''}{(F')^2} \qquad \text{with} \qquad F(0) = \rho_a \qquad \text{and} \qquad F(1) = \rho_b. \tag{7.44}$$

Independent random walkers IRW. As observed in [DG], the independent random walkers model, for which

$$D(\rho) = 1, \quad \sigma(\rho) = 2\rho \tag{7.45}$$

is obtained in the limit as $A = B^{-1} \to 0$ and $C = 1$. Under this choice, see (7.24), all the correlation functions vanish (this obviously reflects the fact that the stationary measure has a product structure, see Proposition 4.5). As $B \to \infty$, one can see from (7.20) that, due to the concavity of the logarithm, the derivative $F'(x)$ is constant. Therefore in this limit the optimal $F(x)$ is given by

$$F(x) = (1-x)\rho_a + x\rho_b \tag{7.46}$$

and one get

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left[\rho(x)\log\left(\frac{\rho(x)}{(1-x)\rho_a + x\rho_b}\right) - \rho(x) + (1-x)\rho_a + x\rho_b\right]. \tag{7.47}$$

KMP model. The expectation of $Q_{i,i+1}(t)$ with respect to the steady state measure μ_{L,T_a,T_b} is now given by

$$\begin{aligned}
\langle Q_{i,i+1}(t)\rangle_{L,T_a,T_b} &= t \cdot \int_0^1 dx\, \langle [x(z_i + z_{i+1}) - z_i] - [(1-x)(z_i+z_{i+1}) - z_{i+1}]\rangle_{L,T_a,T_b} \\
&= t \cdot \langle z_{i+1} - z_i \rangle_{L,T_a,T_b}
\end{aligned} \tag{7.48}$$

then, from (7.1) and (4.26), we get $D(\rho) = 1$. We know that the KMP stationary measure at temperature T is the product of $\texttt{Exponential}(1/T)$. By a duality argument we compute also the mobility and get

$$D(\rho) = 1 \quad \sigma(\rho) = 2\rho^2. \tag{7.49}$$

The KMP model can be, then, obtained (see [DG]) by taking the *unphysical* limit $B \to 0$, $A \to -1$ with $C = 1$. In this limit the first three connected correlations functions (see 7.24) are

$$\begin{aligned} \langle \rho(x) \rangle &= \rho_a(1-x) + \rho_b x \\ \langle \rho(x)\rho(y) \rangle_c &= \frac{(\rho_a - \rho_b)^2}{L} x(1-y) \\ \langle \rho(x)\rho(y)\rho(z) \rangle_c &= -2\frac{(\rho_a - \rho_b)^3}{L^2} x(1-2y)(1-z), \end{aligned} \tag{7.50}$$

which agree with (2.38) of [BDGJL]. Moreover the density large deviation functional that we obtain

$$\mathcal{F}(\{\rho(x)\}) = -\sup_{F(x)} \int_0^1 dx \left[1 - \frac{\rho(x)}{F(x)} + \log \frac{\rho(x)}{F(x)} + \log \frac{F'(x)}{\rho_a - \rho_b} \right] \tag{7.51}$$

agrees with the same function computed in [BGL].

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8 Appendix: Equations for the two points correlations

We provide the linear systems that must be satisfied by the two points correlation functions in the steady state, i.e. $X_{i,\ell} = \langle \eta_i \eta_\ell \rangle$ with $1 \le i \le \ell \le L$. In the following, equations 1),2),3) are obtained by letting act the generator on a couple of sites at distance larger or equal than two, equations 4),5),6) are derived from nearest-neighbouring sites, equations 7),8),9) correspond to the diagonal, equation 10) is obtained from the couple (1,L).

Inclusion/Exclusion walkers: the equations for the inclusion walkers SIP($2k$) and for the exclusion walkers SEP($2j$) are similar, with some relevant change of signs in the two cases; therefore we write them together. With the convention to use upper symbol for inclusion and lower symbol for exclusion in $\pm$ and $\mp$ and with the further convention that $h = k$ for SIP($2k$) and $h = j$ for SEP($2j$), the equations read

1) $X_{i-1,\ell} + X_{i+1,\ell} + X_{i,\ell-1} + X_{i,\ell+1} - 4X_{i,\ell} = 0$ for $i+1 < \ell,\ i > 1,\ \ell < L$

2) $2h(X_{2,\ell} + X_{1,\ell-1} + X_{1,\ell+1}) - (6h \mp \alpha + \gamma)X_{1,\ell} + 2h\alpha x_\ell = 0$ for $\ell > 2$

3) $2h(X_{i,L-1} + X_{i+1,L} + X_{i-1,L}) - (6h + \beta \mp \delta)X_{i,L} + 2h\delta x_i = 0$ for $i < L-1$

4) $hX_{i,i} + hX_{i+1,i+1} + (\mp 1 - 4h)X_{i,i+1} + hX_{i-1,i+1} + hX_{i,i+2} - h(x_i + x_{i+1}) = 0$ for $1 < i < L-1$

5) $2hX_{1,1} + 2hX_{2,2} - (2(3h \pm 1) + (\mp\alpha + \gamma))X_{1,2} + 2hX_{1,3} - 2hx_1 - 2h(1-\alpha)x_2 = 0$

6) $2hX_{L,L} + 2hX_{L-1,L-1} - (2(3h \pm 1) + (\beta \mp \delta))X_{L-1,L} + 2hX_{L-2,L} - 2hx_L - 2h(1-\delta)x_{L-1} = 0$

7) $h(x_{i-1} + 2x_i + x_{i+1}) + (2h \pm 1)X_{i-1,i} - 4hX_{i,i} + (2h \pm 1)X_{i,i+1} = 0$ for $1 < i < L$

8) $2(2h + (\mp\alpha + \gamma))X_{1,1} - 2(2h \pm 1)X_{1,2} - (2h(2\alpha + 1) + \gamma \pm \alpha)x_1 - 2hx_2 - 2h\alpha = 0$

9) $2(2h + (\beta \mp \delta))X_{L,L} - 2(2h \pm 1)X_{L-1,L} - (2h(2\delta + 1) + \beta \pm \delta)x_L - 2hx_{L-1} - 2h\delta = 0$

10) $-(4h + \gamma \mp \delta \mp \alpha + \beta)X_{1,L} + 2hX_{2,L} + 2hX_{1,L-1} + 2h(\delta x_1 + \alpha x_L) = 0$

Brownian energy process BEP$(2k)$: the equations for the BEP$(2k)$ read

1) $X_{i-1,\ell} + X_{i+1,\ell} + X_{i,\ell-1} + X_{i,\ell+1} - 4X_{i,\ell} = 0$ for $i+1 < \ell, \quad i > 1,\ \ell < L$

2) $4k(X_{1,\ell-1} + X_{1,\ell+1} + X_{2,\ell}) - (1 + 12k)X_{1,\ell} + 4kT_a\langle z_\ell\rangle = 0$ for $\ell > 2$

3) $4k(X_{i-1,L} + X_{i+1,L} + X_{i,L-1}) - (12k + 1)X_{i,L} + 4kT_b\langle z_i\rangle = 0$ for $i < L-1$

4) $2kX_{i,i} + 2kX_{i+1,i+1} - 2(4k + 1)X_{i,i+1} + 2k(X_{i-1,i+1} + X_{i,i+2}) = 0$ for $1 < i < L-1$

5) $4k(X_{1,1} + X_{2,2}) - (12k + 5)X_{1,2} + 4kX_{1,3} + 4kT_a\langle z_2\rangle = 0$

6) $4k(X_{L,L} + X_{L-1,L-1}) - (12k + 5)X_{L-1,L} + 4kX_{L-2,L} + 4kT_b\langle z_{L-1}\rangle = 0$

7) $(2k + 1)X_{i-1,i} + (2k + 1)X_{i,i+1} - 4kX_{i,i} = 0$ for $1 < i < L$

8) $2(2k + 1)X_{1,2} - (4k + 1)X_{1,1} + 2(2k + 1)T_a\langle z_1\rangle = 0$

9) $2(2k + 1)X_{L-1,L} - (4k + 1)X_{L,L} + 2(2k + 1)T_b\langle z_L\rangle = 0$

10) $4kT_a\langle z_L\rangle + 4kT_b\langle z_1\rangle - 2(1 + 4k)X_{1,L} + 4k(X_{2,L} + X_{1,L-1}) = 0$

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