

# VARIATIONAL BOUNDS FOR THE GENERALIZED RANDOM ENERGY MODEL

Cristian Giardinà

EURANDOM

P.O. Box 513 - 5600 MB Eindhoven, The Netherlands

e-mail: *giardina@eurandom.tue.nl*

Shannon Starr

UCLA Mathematics Department

Box 951555, Los Angeles, CA 90095-1555

e-mail: *sstarr@math.ucla.edu*

## Abstract

We compute the pressure of the random energy model (REM) and generalized random energy model (GREM) by establishing variational upper and lower bounds. For the upper bound, we generalize Guerra's "broken replica symmetry bounds", and identify the random probability cascade as the appropriate random overlap structure for the model. For the REM the lower bound is obtained, in the high temperature regime using Talagrand's concentration of measure inequality, and in the low temperature regime using convexity and the high temperature formula. The lower bound for the GREM follows from the lower bound for the REM by induction. While the argument for the lower bound is fairly standard, our proof of the upper bound is new.

# 1 Introduction

While the analysis and properties of finite range spin glass systems (like the Edwards-Anderson model) is still a very debated issue, even in the physics community, in recent years there has been a large progress in the mathematical understanding of mean-field models [19]. This advance was triggered by the introduction of a “quadratic interpolation” technique, pioneered in [16] to establish the existence of thermodynamic limit for the Sherrington-Kirkpatrick (SK) model [23] and further developed by Guerra in [15] to prove an upper bound for the pressure which coincides with the Parisi replica symmetry breaking solution of the model. Motivated by the cavity picture, this bound was generalized to a variational bound by the introduction of Random Overlap Structures (ROSt) and associated random weights in [1]. Guerra’s bound is recovered when the weights are chosen from the inhomogeneous Poisson point process studied by Ruelle [22] and the ROSt is a hierarchical one. Recently Talagrand [25] was able to prove that Guerra’s bound is optimal, by showing that the correction term in the bound goes to zero in the thermodynamic limit, thus establishing the rigorous validity of the Parisi solution.

In this paper we solve the Generalized Random Energy Model (GREM) [10, 11] by a simple analysis that partially follows the one developed for the SK model. We first obtain a variational upper bound for the pressure by the definition of the appropriate auxiliary system (i.e., ROSt) for the model. As a guideline in this step we use the basic covariance inequality that was identified in [7]. To show that the upper bound is optimal when the ROSt random weights are chosen according to the Poisson-Dirichlet point process we use a different strategy than the one developed for the SK model. Indeed the corresponding lower bound is easily obtained from a complete control of the high temperature region and convexity of the pressure. To study the high temperature region we propose a new induction argument, which starting from infinite temperature covers all the temperature values up to the critical one.

Despite the recent progress, there is not yet a direct proof of the most prominent property of the Parisi solution, namely *ultrametricity*. A distance  $d$  in a metric space is said to be ultrametric if the standard triangular inequality is replaced by the much stronger in-

equality  $d(x, y) \leq \max\{d(x, y), d(y, z)\}$ . Equivalently, in an ultrametric space all triangles are equilateral or isosceles. The Generalized Random Energy Model (GREM), introduced by Derrida [10] as a model that possesses replica symmetry breaking, is ultrametric by construction, since its Hamiltonian is defined as the sum of independent Gaussian random variables positioned on the branches of a hierarchical tree. As a first step in the direction of achieving a proof of ultrametricity for mean-field models, generalized non-hierarchical models have been considered in [2], where it has been shown that they exhibit GREM-like behaviour. The analysis of the present paper, which sets the GREM model into the general variational scheme developed for the SK model, could be helpful in further studying non-hierarchical models.

The paper is organized as follow. In Section 2 we treat the basic case of the Random Energy Model (REM), where the energy levels are independent Gaussian random variables. This is a necessary warm-up since the GREM model will be treated as a nested succession of REM-like systems. The full analysis is presented in Section 3. In the Appendix we recall some useful concentration of measure estimates.

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## 2 REM

The random energy model (REM) is a statistical mechanical model, where the energy levels are independent and identically distributed Gaussian random variables. More precisely, for a system of size  $N$ , the Hamiltonian is a Gaussian centered family with covariance matrix

$$C_H(\sigma, \sigma') := \mathbb{E}[H_N(\sigma)H_N(\sigma')] = \frac{N}{2} \delta(\sigma, \sigma') \quad (1)$$

where  $\sigma, \sigma' \in \{+1, -1\}^N$  are vectors, whose components are Ising spin variables. The dependence on  $N$  in (1) is such that thermodynamic observables (energy, free energy, etc.)

are extensive in the volume, while the factor  $1/2$  is included as a matter of convention. In this paper we denote by  $X$  a standard Gaussian,  $\mathbb{E}[X] = 0$ ,  $\mathbb{E}[X^2] = 1$ , thus an explicit representation of the Hamiltonian is

$$H_N(\sigma) = \sqrt{\frac{N}{2}} X(\sigma) \quad (2)$$

where  $\{X(\sigma)\}_{\sigma \in \Sigma_N}$  are  $2^N$  i.i.d. copies of the random variable  $X$ . An equivalent representation, more in the spirit of statistical mechanics, is to consider a “lattice”  $\Lambda$  with  $N = |\Lambda|$  sites and a random Hamiltonian

$$H_\Lambda(\sigma) = \sqrt{\frac{N}{2^{N+1}}} \sum_{X \subseteq \Lambda} J_X \sigma_X \quad (3)$$

defined on the spin configurations  $\sigma : \Lambda \rightarrow \{+1, -1\}$ . In (3) the  $J_X$ ’s are a family of i.i.d. Gaussian random variables, with  $\mathbb{E}[J_X^2] = 1$ , and  $\sigma_X := \prod_{i \in X} \sigma_i$  for each  $X \subseteq \Lambda$ . Henceforth, all Gaussian random variables will be understood to have expectation equal to 0.

The main quantity we are going to study is the quenched *pressure*. We denote by  $\Sigma_N = \{-1, +1\}^N$  the space of all possible spin configurations. For a finite system we define the *random* partition function

$$Z_N(\beta) = \sum_{\sigma \in \Sigma_N} e^{-\beta H_N(\sigma)}, \quad (4)$$

and the *quenched* pressure

$$P_N(\beta) = \mathbb{E} \left[ \frac{1}{N} \ln Z_N(\beta) \right]. \quad (5)$$

We will be interested in the thermodynamic limit

$$P(\beta) = \lim_{N \rightarrow \infty} P_N(\beta) \quad (6)$$

In the following we will sometimes drop the  $N$ -dependence in the Hamiltonian  $H(\sigma)$  in order to alleviate notation. We will also introduce additional randomness by considering an auxiliary system which is coupled to the Hamiltonian. We will denote by  $\mathbb{E}[\cdot]$  the expectation with respect to all random variables involved.

## 2.1 Upper bound

We start by recalling the quadratic interpolation technique.

**Lemma 2.1** *Let  $H(\sigma)$  be a Gaussian family, indexed by  $\sigma \in \Sigma_N$ , with covariance  $C_H(\sigma; \sigma')$ . Let  $\alpha \in \mathcal{A}$  be an index ranging over the set  $\mathcal{A}$ , and let  $K(\alpha)$  and  $V(\sigma, \alpha)$  be Gaussian random variables, independent of  $H(\sigma)$  and of each other, with covariances  $C_K(\alpha; \alpha')$  and  $C_V(\sigma, \alpha; \sigma', \alpha')$ , respectively. Suppose that*

$$C_H(\sigma; \sigma') + C_K(\alpha; \alpha') \geq C_V(\sigma, \alpha; \sigma', \alpha') \quad (7)$$

for all  $\sigma, \sigma' \in \Sigma_N$  and  $\alpha, \alpha' \in \mathcal{A}$ , and suppose that

$$C_H(\sigma; \sigma) + C_K(\alpha; \alpha) = C_V(\sigma, \alpha; \sigma, \alpha) \quad (8)$$

for every  $\sigma \in \Sigma_N$  and  $\alpha \in \mathcal{A}$ . Moreover, suppose that there is a random weight  $w : \mathcal{A} \rightarrow [0, \infty)$  such that, almost surely,  $\sum_{\alpha \in \mathcal{A}} w(\alpha)$  is strictly positive and finite. Then,

$$\begin{aligned} P_N(\beta) &\leq \mathbb{E} \left[ \frac{1}{N} \ln \frac{\sum_{\sigma, \alpha} w(\alpha) e^{-\beta V(\sigma, \alpha)}}{\sum_{\alpha} w(\alpha) e^{-\beta K(\alpha)}} \right] \\ &= \mathbb{E} \left[ \frac{1}{N} \ln \sum_{\sigma, \alpha} w(\alpha) e^{-\beta V(\sigma, \alpha)} \right] - \mathbb{E} \left[ \frac{1}{N} \ln \sum_{\alpha} w(\alpha) e^{-\beta K(\alpha)} \right], \end{aligned} \quad (9)$$

as long as the right-hand-side is well-defined (i.e., not  $\infty - \infty$ ).

**Proof:** We refer to [16, 1] for full details. Here we only recall the basic idea. For  $t \in [0, 1]$  define an interpolating Hamiltonian

$$\tilde{H}(\sigma, \alpha; t) = \sqrt{1-t} [H(\sigma) + K(\alpha)] + \sqrt{t} V(\sigma, \alpha). \quad (10)$$

and an associated random partition function

$$Z_{N,t}(\beta) = \sum_{\sigma, \alpha} w(\alpha) e^{-\beta \tilde{H}(\sigma, \alpha; t)}. \quad (11)$$

Let  $\Omega_{N,\beta,t}$  denote expectation with respect to the multiple-replica product measure, where the weight for a configuration  $(\sigma, \alpha)$  of a generic copy is given by Gibbs measure associated

to  $\tilde{H}(\sigma, \alpha)$  times a generic weights  $w(\alpha)$ . In particular, for a function  $f(\sigma, \alpha, \sigma', \alpha')$  of two replicas, one has

$$\Omega_{N,\beta,t}\{f(\sigma, \alpha; \sigma', \alpha')\} = \sum_{\sigma,\alpha} \sum_{\sigma',\alpha'} w(\alpha)w(\alpha') \frac{e^{-\beta[\tilde{H}(\sigma,\alpha;t)+\tilde{H}(\sigma',\alpha';t)]}}{Z_{N,t}^2} f(\sigma, \alpha; \sigma', \alpha').$$

Then one actually has (because of the equality along the diagonal Eq. (8))

$$\begin{aligned} \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\sigma,\alpha} w(\alpha) e^{-\beta V(\sigma,\alpha)} \right] - \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} w(\alpha) e^{-\beta K(\alpha)} \right] - P_N(\beta) \\ = \frac{\beta^2}{2} \int_0^1 \mathbb{E} [\Omega_{N,\beta,t} \{ C_H(\sigma; \sigma') + C_K(\alpha; \alpha') - C_V(\sigma, \alpha; \sigma', \alpha') \}] dt. \end{aligned} \quad (12)$$

This is proved by differentiating the quantity

$$\mathbb{E} [N^{-1} \ln \sum_{\alpha,\sigma} w(\alpha) e^{-\beta \tilde{H}_N(\sigma,\alpha;t)}],$$

with respect to  $t$  and using the generalized Wick's rule. Because of (7), the right hand side of (12) is obviously positive and equation (9) follows.  $\square$

**Remark 1** *The same basic argument works to bound  $\mathbb{E}[F(Z_N(\beta))]$  for other functions such as  $F(z) = z^a$ . Remarkably, this inequality was discovered some time before [16], in the context of general properties of Gaussians [18]. Of course it is more than just that inequality which makes [16] so important: it is also the application.*

**Remark 2** *An identity such as (12) is usually called a sum-rule. The process  $K(\alpha)$  has to be thought of as a large reservoir which acts on the original system  $H(\sigma)$  through the interaction  $V(\sigma, \alpha)$ .*

We are going to use the previous lemma to establish an optimal upper bound for the REM model. A key element is to choose the correct formula for the random weight  $w(\alpha)$ . The correct formula for mean field spin glasses seems to generally be given by Ruelle's random probability cascade. For the REM, it is given by a single level of that, which is sometimes called the Poisson-Dirichlet process<sup>1</sup>. Let us give a brief description of this (the unnormalized version) in order to facilitate the following proposition.

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<sup>1</sup>See [21] but also see [22]. For a rather more abstract version, see [3].

Given  $0 < m < 1$  consider the Poisson point process on  $(0, \infty)$  with intensity measure equal to  $mw^{-m-1}dw$ . Almost surely, the points can be labelled  $\{w_1, w_2, \dots\}$  with  $w_1 > w_2 > \dots > 0$ . Moreover,  $\sum_{\alpha=1}^{\infty} w_{\alpha}$  is strictly positive and finite, almost surely. The distribution of  $\{w_{\alpha}\}_{\alpha}$  has a remarkable invariance property: If  $(f_1, f_2, \dots)$  are i.i.d. copies of the random variable,  $f$ , which are assumed to be independent of  $\{w_{\alpha}\}_{\alpha}$ , then (modulo permutations) the distribution of  $\{e^{f_{\alpha}}w_{\alpha}\}_{\alpha}$  is the same as  $\{cw_{\alpha}\}_{\alpha}$  where  $c$  is the nonrandom number  $c = (\mathbb{E}[e^{mf}])^{1/m}$ . This is easily proved using the generalized Laplace transform. (For a proof see, for example, reference [24], page 481.)

One may note that instead of considering the Poisson point process  $\{w_{\alpha}\}_{\alpha}$  with intensity equal to  $mw^{-m-1}dw$ , one could instead consider  $w_{\alpha} = e^{-y_{\alpha}/m}$  for some  $\{y_{\alpha}\}_{\alpha}$ . Then  $-\infty < y_1 < y_2 < \dots < \infty$  is a Poisson point process on  $\mathbb{R}$  with intensity measure  $e^{-y}dy$  (independent of  $m$ ). One thinks of the  $y_{\alpha}$ 's as free energies. In this notation  $m$  is explicit. Henceforth  $\{y_{\alpha}\}_{\alpha}$  will refer to the point process just described.

A consequence of the invariance property mentioned above is

$$\sum_{\alpha=1}^{\infty} e^{-y_{\alpha}/m} \exp(f_{\alpha}) \stackrel{\mathcal{D}}{=} \mathbb{E}[\exp(mf)]^{1/m} \sum_{\alpha=1}^{\infty} e^{-y_{\alpha}/m}, \quad (13)$$

which will be useful.

**Proposition 2.2** *Let  $C_H(\sigma, \sigma')$  be given by Eq. (1). Choose  $w(\alpha) = \exp[-y(\alpha)/m]$  for  $0 < m < 1$ . For each  $b \geq 1$  let*

$$C_K(\alpha, \alpha') = (b-1) \frac{N}{2} \delta(\alpha, \alpha') \quad (14)$$

$$C_V(\sigma, \alpha; \sigma', \alpha') = b \frac{N}{2} \delta(\sigma, \sigma') \delta(\alpha, \alpha'). \quad (15)$$

*Then one obtains the optimal upper bound for the REM,*

$$P_N(\beta) \leq \inf_{0 < m < 1} \left[ \frac{1}{4} m \beta^2 + \frac{1}{m} \ln 2 \right] \quad (16)$$

**Proof:** We note that Lemma 2.1 is applicable because

$$b \delta(\sigma, \sigma') \delta(\alpha, \alpha') \leq \delta(\sigma, \sigma') + (b-1) \delta(\alpha, \alpha') \quad (17)$$

We compute separately the two terms in Eq.(9). For the first one, due to Eq. (15), we have

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\sigma, \alpha} w(\alpha) \exp[-\beta V(\sigma, \alpha)] \right] = \quad (18)$$

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] \sum_{\sigma} \exp \left[ -\beta \sqrt{\frac{bN}{2}} X(\sigma, \alpha) \right] \right] = \quad (19)$$

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] Z_N(\beta\sqrt{b}; \alpha) \right] \quad (20)$$

where  $Z_N(\beta\sqrt{b}; \alpha)$  are independent copies (labeled by the  $\alpha$ 's) of the random variable  $Z_N(\beta\sqrt{b}) = \sum_{\sigma} \exp \left[ -\beta \sqrt{\frac{bN}{2}} X(\sigma) \right]$ . By applying the invariance property of Eq.(13) to Eq.(20) with  $\exp[f_{\alpha}] = Z_N(\beta\sqrt{b}; \alpha)$  we obtain

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\sigma, \alpha} w(\alpha) \exp[-\beta V(\sigma, \alpha)] \right] = \quad (21)$$

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] \right] + \frac{1}{mN} \ln \mathbb{E} \left[ Z_N^m(\beta\sqrt{b}) \right]. \quad (22)$$

Then we consider the second term in Eq.(9). Taking into account the choice (14) we have

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] \exp[-\beta K(\alpha)] \right] = \quad (23)$$

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] \exp \left[ -\beta \sqrt{\frac{(b-1)N}{2}} X_{\alpha} \right] \right]. \quad (24)$$

Using again the invariance property Eq.(13) with  $\exp[f_{\alpha}] = \exp \left[ -\beta \sqrt{\frac{(b-1)N}{2}} X_{\alpha} \right]$  and computing the average we obtain

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] \exp[-\beta K(\alpha)] \right] = \quad (25)$$

$$\frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} \exp \left[ \frac{y(\alpha)}{m} \right] \right] + \beta^2 \frac{(b-1)m}{4}. \quad (26)$$

Putting together Eq.(22) and (26) we obtain:

$$P_N(\beta) \leq \frac{1}{mN} \ln \mathbb{E} \left[ Z_N^m(\beta\sqrt{b}) \right] - \beta^2 \frac{(b-1)m}{4}. \quad (27)$$



Now we use the simple fact that

$$Z_N^m(\beta\sqrt{b}) \leq Z_N(m\beta\sqrt{b}). \quad (28)$$

This is a general fact in statistical mechanics: since the entropy is positive by definition, the free energy is increasing in  $\beta$ . Indeed, considering  $f_N(\beta) = -\frac{1}{N\beta} \ln(Z_N(\beta))$ , the *random* free energy, one immediately checks that  $f'_N(\beta) = \frac{1}{\beta} (u_N(\beta) - f_N(\beta)) = \frac{1}{\beta^2} s_N(\beta) \geq 0$ , where  $u_N(\beta)$  is the *random* internal energy, and  $s_N(\beta)$  is the *random* entropy. Therefore, for any  $0 < m \leq 1$ , we have:

$$-\frac{1}{Nm\beta} \ln(Z_N(m\beta)) \leq -\frac{1}{N\beta} \ln(Z_N(\beta)), \quad (29)$$

which is equivalent to (28) when we replace  $\beta$  by  $\beta\sqrt{b}$ . By inserting Eq.(28) into Eq.(27) it is now easy to compute the expectation  $\mathbb{E} \left[ Z_N(m\beta\sqrt{b}) \right]$  and we arrive at the upper bound

$$P_N(\beta) \leq \frac{1}{4}m\beta^2 + \frac{1}{m} \ln 2. \quad (30)$$

Note that there is not anymore dependence on  $b$  in the bound. Finally the optimal bound is obtained by minimization in  $m$  which yields the (16) and completes the proof.  $\square$

**Remark 3** *It is interesting to note that for the REM (and for the REM only) the main result of quadratic interpolation can be viewed as an implementation of Jensen's inequality:*

$$\mathbb{E}[\ln(Z_N(\beta))] = \mathbb{E} \left[ \frac{1}{m} \ln Z_N^m(\beta) \right] = \frac{1}{m} \mathbb{E}[\ln Z_N^m(\beta)] \leq \frac{1}{m} \ln \mathbb{E}[Z_N^m(\beta)]. \quad (31)$$

*This holds for general  $m > 0$ , but one needs  $m \leq 1$  to apply (28). If one uses the “sum-rule” then one can get an explicit form for the error coming from Jensen's inequality in this case.*

**Remark 4** *The inequality of (16) takes two different forms depending on whether  $\beta$  is greater than or less than  $\beta_c := 2\sqrt{\ln(2)}$ . For  $\beta > \beta_c$ , the right-hand-side of (16) is optimized at  $m = \beta_c/\beta$ . For  $\beta < \beta_c$  the infimum over  $0 < m < 1$  is attained by a limit  $m \rightarrow 1$ . Therefore, one has*

$$P_N(\beta) \leq \mathcal{Q}(\beta), \quad (32)$$

where

$$\mathcal{Q}(\beta) = \left\{ \begin{array}{ll} \frac{1}{4}\beta^2 + \ln(2) & \text{for } \beta < \beta_c \\ \beta\sqrt{\ln(2)} & \text{for } \beta \geq \beta_c \end{array} \right\} = \left\{ \begin{array}{ll} \frac{1}{4}(\beta^2 + \beta_c^2) & \text{for } \beta < \beta_c \\ \frac{1}{2}\beta\beta_c & \text{for } \beta \geq \beta_c \end{array} \right\} \quad (33)$$

Note that the two pieces match at  $\beta = \beta_c$ .

## 2.2 Lower bound

The main new result in this paper is the adaptation of the quadratic interpolation method to obtain an asymptotically sharp upper bound on  $P_N(\beta)$ , as we just considered above. One should view the result of the previous section as an analogue, for the REM, of Guerra's bounds for the SK model in [15]. On the other hand, for the REM, unlike for the SK model, there are easy proofs of the same lower bound in the  $N \rightarrow \infty$  limit. The exact formula for the pressure of the random energy model is well-known. Derrida calculated it when he introduced the model in [10] while a mathematically rigorous version of his argument is included in [4, 20]. There are also proofs which rely more on large-deviation theory, such as [13, 12]<sup>2</sup>. We also encourage the reader to see the more recent, deep analysis of [5, 6]. For completeness, we will also present a proof of the lower bound.

Since  $P_N(\beta) \leq \mathcal{Q}(\beta)$  for all  $N$ , it follows that  $P(\beta) \leq \mathcal{Q}(\beta)$  in the limit  $N \rightarrow \infty$ . We want to show the opposite is also true, to establish that  $P(\beta) = \mathcal{Q}(\beta)$  for all  $\beta \geq 0$ . The key to obtaining the lower bound is to understand the high temperature region,  $\beta < \beta_c$ .

### Proposition 2.3

$$P(\beta) = \mathcal{Q}(\beta) \quad \text{for } \beta \geq \beta_c. \quad (34)$$

The proof of this result is provided in the next subsection, while here we stress its consequences. Stated otherwise, the upper bound of Proposition 2.2 saturates in the  $N \rightarrow \infty$  limit, at least when  $\beta \leq \beta_c$ . It is a remarkable fact that this high-temperature result gives the sharpness of the upper bound also in the low-temperature region as follows.

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<sup>2</sup>In [13], there is also a conjecture for the more general quantity  $N^{-1} \ln E[Z_N(\beta)^a]$  for  $a \in \mathbb{R}$ . This is substantiated in the preliminary section of [26].

**Corollary 2.4** For  $\beta \in [0, +\infty)$

$$P(\beta) = \mathcal{Q}(\beta). \quad (35)$$

**Proof:** It is a basic fact, easily seen from the definition (5), that  $P_N(\beta)$  is convex in  $\beta$  for each  $N$ . Therefore, the limiting function  $P(\beta)$  is also convex. Hence, for any  $\beta_0$  and any  $\beta \geq \beta_0$ , we have

$$P(\beta) \geq P(\beta_0) + (\beta - \beta_0)DP(\beta_0), \quad (36)$$

where  $D$  is any convex combination of the left-handed and right-handed derivatives, which we denote  $D_-$  and  $D_+$ , respectively. We now take  $\beta_0 \uparrow \beta_c$ . Since we know from Proposition 2.3 that  $P(\beta) = \mathcal{Q}(\beta)$  for  $\beta < \beta_c$ , we easily calculate  $\lim_{\beta_0 \uparrow \beta_c} P(\beta_0) = 2 \ln(2)$  while  $\lim_{\beta_0 \uparrow \beta_c} D_-P(\beta_0) = \lim_{\beta_0 \uparrow \beta_c} \mathcal{Q}'(\beta_0) = \sqrt{\ln(2)}$ . Putting this together completes the proof.  $\square$

## 2.3 High temperature region

The proof of Proposition 2.3 will be obtained through a sequence of lemmata. The crux of the argument is standard. (C.f., [24] Proposition 1.1.5). We start with the following result, which is another variational calculation.

**Lemma 2.5** Let  $\Omega_{\beta,N}$  refer to the (expectation associated to the) random probability measure on  $\Sigma_N \times \Sigma_N$  specified by

$$\Omega_{\beta,N}\{f(\sigma, \sigma')\} := Z_N^{-2}(\beta) \sum_{\sigma, \sigma' \in \Sigma_N} e^{-\beta H(\sigma)} e^{-\beta H(\sigma')} f(\sigma, \sigma').$$

For  $0 \leq \beta \leq \beta_c$  we have

$$\frac{1}{N} \ln \mathbb{E}[Z_N(\beta) \Omega_{\beta,N}\{\delta(\sigma, \sigma')\}] \leq \frac{\beta \beta_c}{2}. \quad (37)$$

**Proof:** Note that

$$\Omega_{\beta,N}\{\delta(\sigma, \sigma')\} = \frac{\sum_{\sigma, \sigma' \in \Sigma_N} e^{-\beta H(\sigma)} e^{-\beta H(\sigma')} \delta(\sigma, \sigma')}{Z_N^2(\beta)} = \frac{Z_N(2\beta)}{Z_N^2(\beta)}. \quad (38)$$

By (28), we know that for  $0 < m < 1$

$$Z_N(2\beta) \leq Z_N^{1/m}(2m\beta). \quad (39)$$

Using Hölder's inequality with  $p = 1/m$ ,  $q = 1/(1 - m)$  and  $\frac{1}{p} + \frac{1}{q} = 1$

$$Z_N(2m\beta) \leq Z_N^m(\beta) Z_N^{1-m}\left(\frac{m}{1-m}\beta\right). \quad (40)$$

Therefore,

$$Z_N(\beta)\Omega_{\beta,N}\{\delta(\sigma, \sigma')\} = \frac{Z_N(2\beta)}{Z_N(\beta)} \leq \frac{Z_N^{1/m}(2m\beta)}{Z_N(\beta)} \leq Z_N^{(1-m)/m}\left(\frac{m}{1-m}\beta\right).$$

Since  $m/(1 - m)$  can take any positive value as  $m$  ranges over  $(0, 1)$  this means

$$Z_N(\beta)\Omega_{\beta,N}\{\delta(\sigma, \sigma')\} \leq Z_N^{1/r}(r\beta) \quad (41)$$

for every  $r \in [0, +\infty)$ . Moreover, for  $r \geq 1$  we can use Jensen's inequality to obtain

$$\mathbb{E}[Z_N^{1/r}(r\beta)] \leq (\mathbb{E}[Z_N(r\beta)])^{1/r} = \exp\left(N\left[\frac{\beta_c^2}{4r} + \frac{r\beta^2}{4}\right]\right). \quad (42)$$

It is easy to see that the optimal value is  $r = \beta_c/\beta$ , which does satisfy the constraint  $r \geq 1$  because of the hypothesis  $\beta \leq \beta_c$ . Choosing this  $r$  and putting (41) and (42) together yields (37).  $\square$

A second useful estimate is the following concentration of measure property.

**Lemma 2.6** *For any  $\beta$ , and any  $t > 0$ ,*

$$\mathbb{P}\{|N^{-1} \ln Z_N(\beta) - P_N(\beta)| \geq \beta t\} \leq 2e^{-Nt^2/2}.$$

The analogous result for the SK model is Corollary 2.2.5 of [24]. For completeness we will give a proof of Lemma 2.6 in Appendix A and, particularly, show that a straightforward generalization of Talagrand's proof applies equally well to the REM, the GREM, the SK model, and all the  $p$ -spin models in between.

The proof of Proposition 2.3 will essentially follow from the next result, which we prove first.

**Lemma 2.7** *For any  $0 \leq \beta < \beta_c$ ,*

$$\limsup_{N \rightarrow \infty} \sup_{0 \leq \beta' \leq \beta} \frac{1}{N} \ln \mathbb{E}[\Omega_{\beta',N}\{\delta(\sigma, \sigma')\}] < 0. \quad (43)$$

**Proof:** The proof will be obtained by induction. Let us define the succession of temperatures given by  $\beta_0 = 0$ ,  $\beta_{n+1} = g(\beta_n)$  for  $n \in \mathbb{N}$ , where  $g$  is a definite function. As we will see, we can choose

$$g(\beta) = \beta + a\beta_c[1 - (\beta/\beta_c)]^2 \quad \text{for any } 0 < a < 1/2. \quad (44)$$

and it will follow that  $\beta_n \uparrow \beta_c$  as  $n \rightarrow \infty$ .

We first note that (43) is true for  $\beta = \beta_0 = 0$ , because one has  $\mathbb{E}[\Omega_{0,N}\{\delta(\sigma, \sigma')\}] = 2^{-N}$ . For the induction step, we will prove that if (43) is true for  $\beta \in [0, \beta_n]$ , then it is also true for every  $\beta \in [0, \beta_{n+1}]$ . Then, since  $\beta_n \uparrow \beta_c$  the statement of Lemma 2.7 follows.

To prove the induction step, we first observe that

$$\frac{d}{d\beta} P_N(\beta) = \frac{\beta}{2} (1 - \mathbb{E}[\Omega_{N,\beta}\{\delta(\sigma, \sigma')\}]) \geq 0. \quad (45)$$

Indeed, this is a simple calculation using the generalized Wick's rule. Since  $Z_N(0) = 2^N$  (deterministically) we have  $P_N(0) = \ln(2)$ . Then the sum rule

$$P_N(\beta) = \ln(2) + \frac{\beta^2}{4} - \int_0^\beta \frac{\beta'}{2} \mathbb{E}[\Omega_{N,\beta'}\{\delta(\sigma, \sigma')\}] d\beta' \quad (46)$$

follows.

Suppose now that (43) is true for  $\beta \in [0, \beta_n]$ . Consider a generic  $\beta > \beta_n$ , not necessarily smaller than  $\beta_{n+1}$ , and for  $t > 0$  let  $A_N(\beta, t)$  be the event in Lemma 2.6:

$$A_N(\beta, t) = \{|N^{-1} \ln Z_N(\beta) - P_N(\beta)| \geq \beta t\}.$$

On  $A_N(\beta, t)^c$  we have  $Z_N(\beta) \geq e^{N[P_N(\beta) - \beta t]}$ . Let us employ the following shorthand: given a set  $A$ , let  $I_A$  denote the indicator of  $A$  and let us denote  $\mathbb{E}[I_A X]$  by  $\mathbb{E}[X, A]$  for each and every random variable  $X$ . Then we conclude that

$$\begin{aligned} \mathbb{E}[\Omega_{\beta,N}\{\delta(\sigma, \sigma')\}, A_N(\beta, t)^c] &\leq \mathbb{E}\left[\frac{Z_N(\beta)}{e^{N[P_N(\beta) - \beta t]}} \Omega_{\beta,N}\{\delta(\sigma, \sigma')\}, A_N(\beta, t)^c\right] \\ &\leq e^{-N[P_N(\beta) - \beta t]} \mathbb{E}[Z_N(\beta) \Omega_{\beta,N}\{\delta(\sigma, \sigma')\}]. \end{aligned}$$

Therefore

$$\frac{1}{N} \ln \mathbb{E}[\Omega_{\beta,N}\{\delta(\sigma, \sigma')\}, A_N(\beta, t)^c] \leq -P_N(\beta) + \frac{1}{2}\beta\beta_c + \beta t$$

follows from Lemma 2.5.

Since we assumed  $\beta > \beta_n$ , a lower bound for  $P_N(\beta)$  is given by  $P_N(\beta_n)$  and a lower bound for  $P_N(\beta_n)$  is given by  $\frac{1}{4}[\beta_c^2 + \beta_n^2] - o(1)$ , using Eq. (46) and the induction hypothesis, where  $o(1)$  represents a quantity whose limit is 0 when  $N \rightarrow \infty$ . One thing which is important is that while  $o(1)$  does depend on  $\beta_n$ , it is independent of  $\beta > \beta_n$ . Therefore,

$$\frac{1}{N} \ln \mathbb{E}[\Omega_{\beta,N}\{\delta(\sigma, \sigma')\}, A_N(\beta, t)^c] \leq -\frac{1}{4}(\beta_c - \beta_n)^2 + \frac{1}{2}\beta_c[\beta - \beta_n] + \beta t - o(1). \quad (47)$$

On the other hand, one always has  $0 \leq \Omega_{\beta,N}\{\delta(\sigma, \sigma')\} \leq 1$ . Hence,

$$\mathbb{E}[\Omega_{\beta,N}\{\delta(\sigma, \sigma')\}, A_N(\beta, t)] \leq \mathbb{E}[1, A_N(\beta, t)] = \mathbb{P}(A_N(\beta, t)).$$

So, by Lemma 2.6,

$$\frac{1}{N} \ln \mathbb{E}[\Omega_{\beta,N}\{\delta(\sigma, \sigma')\}, A_N(\beta, t)] \leq -\frac{1}{2}t^2 + N^{-1} \ln(2). \quad (48)$$

Putting equations (47) and (48) together we obtain

$$\begin{aligned} & \frac{1}{N} \ln \mathbb{E}[\Omega_{\beta,N}\{\delta(\sigma, \sigma')\}] \\ & \leq N^{-1} \ln(2) + \max \left\{ -\frac{1}{2}t^2 + N^{-1} \ln(2), -\frac{1}{4}(\beta_c - \beta_n)^2 + \frac{1}{2}\beta_c[\beta - \beta_n] + \beta t - o(1) \right\}. \end{aligned} \quad (49)$$

If we now take

$$\beta < \beta_n + \frac{1}{2}\beta_c[1 - (\beta_n/\beta_c)]^2$$

then it is clear that by choosing  $t$  positive, but small enough, we will have a strictly negative limsup of the left hand side of (49) as  $N \rightarrow \infty$ . Choosing any  $0 < a < 1/2$ , let us take  $g(\beta) = \beta + a\beta_c[1 - (\beta/\beta_c)]^2$ . Then for  $\beta_{n+1} = g(\beta_n)$ , we have proved the induction step: for  $\beta$  in the range  $[0, \beta_{n+1}]$ , inequality (43) also holds.  $\square$

We complete this section with the proof of Proposition 2.3.

**Proof:** By Lemma 2.7, the integrand of the third term in the right hand side of Eq. (46) approaches 0 uniformly as  $N \rightarrow \infty$ , as long as  $0 \leq \beta < \beta_c$ . This gives the desired result.

$\square$

### 3 The Generalized Random Energy Model

In this section we extend the method developed in the previous section to treat the GREM. This is essentially a “correlated random energy model” on a hierarchical graph – that is, a tree.

#### 3.1 Set-up and Basics

The GREM is a family of models, taking various parameters for the definition. Let  $n \in \mathbb{N}_+$  be an integer, equal to the number of levels in the hierarchical tree. Let  $K_1, \dots, K_n$  be positive integers such that  $K_1 + K_2 + \dots + K_n = N$ , where  $N$  is the system size. Also, let  $a_1, \dots, a_n$  be real numbers such that  $0 < a_i$  for  $i = 1, \dots, n$  and  $a_1 + a_2 + \dots + a_n = 1$ .

**Definition 3.1** *Given  $N \in \mathbb{N}$  and  $\sigma \in \Sigma_N$ , for  $i = 1, \dots, n$ , let  $\pi_i(\sigma)$  be the canonical projection over the subset  $\Sigma_{K_i}$  generated by the lexicographical partition  $\mathcal{P}$  of the coordinates  $(\sigma_1, \dots, \sigma_N)$  into the first  $K_1$  coordinates, the successive  $K_2$  coordinates and so on up to the last  $K_n$  coordinates. Namely,  $\Sigma_N = \Sigma_{K_1} \times \dots \times \Sigma_{K_n}$ ,  $\otimes_{i=1}^n \pi_i = 1_{\Sigma_N}$  and  $\pi_i(\sigma) = (\sigma_{K_1+\dots+K_{i-1}+1}, \dots, \sigma_{K_1+\dots+K_i})$ .*

Then the GREM Hamiltonian is a family of Gaussian random variables having the covariance

$$\mathbb{E}[H_N(\sigma)H_N(\sigma')] = \frac{N}{2} \sum_{i=1}^n a_i \prod_{j=1}^i \delta(\pi_j(\sigma), \pi_j(\sigma')) \quad (50)$$

An explicit form of GREM Hamiltonian is

$$H_N(\sigma) = \sqrt{\frac{N}{2}} \sum_{i=1}^n \sqrt{a_i} X(\pi_1(\sigma), \dots, \pi_i(\sigma)). \quad (51)$$

where, for each  $i = 1, \dots, n$ , the family of random variables  $\{X(\pi_1(\sigma), \dots, \pi_i(\sigma))\}_{\sigma \in \Sigma_N}$  are  $2^{K_1+K_2+\dots+K_i}$  i.i.d. Gaussians, and each family is independent of the others.

**Remark 5** *The Hamiltonian (51) corresponds to a tree with furcation that at each level is a power of two. We will stick to this case to simplify the notation, while the more general case of arbitrary furcation (with the constraint of having approximately  $2^N$  leaves*

in the last layer) is completely equivalent in the thermodynamic limit. In order to make statements that apply in the limit, we will consider sequences of  $N$ 's and  $K_1, \dots, K_n$  such that there are rational numbers  $\kappa_1, \dots, \kappa_n$ , all nonnegative, and summing to 1, with  $K_i = N\kappa_i$  for each  $i = 1, \dots, n$ .

We now prove the variational expression for the pressure of the GREM. The strategy is to apply the results obtained for the REM model in the previous section at each level in the hierarchy. In order to denote the dependence on the parameters  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_n)$ , let us write the GREM pressure as

$$P_N^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) = \frac{1}{N} \mathbb{E} [\ln Z_N(\beta; \mathbf{a}, \boldsymbol{\kappa})]. \quad (52)$$

and its thermodynamic limit as  $P^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) := \lim_{N \rightarrow \infty} P_N^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa})$ .

## 3.2 Upper bound

**Proposition 3.2** *Consider the GREM model, for which  $C_H(\sigma, \sigma')$  is given by (50). For an index  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{A}^n$  let the random weights  $w(\alpha)$  be given by*

$$w(\alpha_1, \dots, \alpha_n) = \exp \left[ \frac{y(\alpha_1)}{m_1} \right] \exp \left[ \frac{y(\alpha_1, \alpha_2)}{m_2} \right] \cdots \exp \left[ \frac{y(\alpha_1, \dots, \alpha_n)}{m_n} \right] \quad (53)$$

where the Poisson point processes is now a cascade with intensity measure  $e^{-y} dy$ . Namely,  $y(\alpha_1)$  is the usual PPP, then for each given  $\alpha_1$ ,  $y(\alpha_1, \alpha_2)$  is an independent copy (labelled by  $\alpha_1$ ) of the PPP, ... and so on up to  $y(\alpha_1, \dots, \alpha_n)$  which, for each given  $\alpha_1, \dots, \alpha_{n-1}$ , is an independent copy of the PPP (labelled by  $\alpha_1, \dots, \alpha_{n-1}$ ). We also choose a sequence  $0 < m_1 \leq m_2 \leq m_n < 1$  and

$$C_K(\alpha, \alpha') = (b-1) \frac{N}{2} \sum_{i=1}^n a_i \prod_{j=1}^i \delta(\alpha_j, \alpha'_j) \quad (54)$$

$$C_V(\sigma, \alpha, \sigma', \alpha') = b \frac{N}{2} \sum_{i=1}^n a_i \prod_{j=1}^i \delta(\pi_j(\sigma), \pi_j(\sigma')) \delta(\alpha_j, \alpha'_j), \quad (55)$$

where  $b$  is a real number such that  $b > 1$ . Then we obtain the optimal upper bound:

$$P_N^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) \leq \inf_{0 < m_1 \leq \dots \leq m_n < 1} \sum_{i=1}^n \left[ \frac{\kappa_i}{m_i} \ln(2) + \frac{\beta^2}{4} m_i a_i \right]. \quad (56)$$



**Proof:** It will be along the lines of the proof of Proposition 2.2. Lemma 2.1 is applicable because for each  $i = 1, \dots, n$  one has

$$b \prod_{j=1}^i \delta(\pi_j(\sigma), \pi_j(\sigma')) \delta(\alpha_j, \alpha'_j) \leq \prod_{j=1}^i \delta(\pi_j(\sigma), \pi_j(\sigma')) + (b-1) \prod_{j=1}^i \delta(\alpha_j, \alpha'_j) \quad (57)$$

For the first term of Eq.(9), using Eq. (55), we have

$$\begin{aligned} \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\sigma, \alpha} w(\alpha) \exp[-\beta V(\sigma, \alpha)] \right] &= \quad (58) \\ \frac{1}{N} \mathbb{E} \left( \ln \sum_{\sigma, \alpha} \exp \left[ \frac{y(\alpha_1)}{m_1} \right] \exp \left[ \frac{y(\alpha_1, \alpha_2)}{m_2} \right] \cdots \exp \left[ \frac{y(\alpha_1, \dots, \alpha_n)}{m_n} \right] \right. \\ &\quad \left. \exp \left[ -\beta \sqrt{\frac{bN}{2}} \sum_{i=1}^n \sqrt{a_i} X(\pi_1(\sigma), \pi_2(\sigma), \dots, \pi_i(\sigma), \alpha_1, \alpha_2, \dots, \alpha_i) \right] \right) \end{aligned}$$

Since the sum over configurations  $\sigma \in \Sigma_N, \alpha \in \mathcal{A}^n$  can be decomposed into  $n$  sums over each subset  $\pi_i(\sigma) \in \Sigma_{K_i}, \alpha_i \in \mathcal{A}$  for  $i = 1, \dots, n$ , the invariance property (13) can now be applied telescopically, starting at the  $n$ th level and tracing back up to the first level. After this simplification we obtain

$$\begin{aligned} \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\sigma, \alpha} w(\alpha) \exp[-\beta V(\sigma, \alpha)] \right] &= \quad (59) \\ \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} w(\alpha) \right] + \sum_{i=1}^n \frac{1}{m_i N} \ln \mathbb{E} \left[ Z_{K_i}^{m_i}(\beta \sqrt{b a_i}) \right] &\leq \\ \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} w(\alpha) \right] + \sum_{i=1}^n \frac{1}{m_i N} \ln \mathbb{E} \left[ Z_{K_i}(m_i \beta \sqrt{b a_i}) \right] &= \\ \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} w(\alpha) \right] + \sum_{i=1}^n \left[ \frac{\kappa_i}{m_i} \ln(2) \right] + \frac{\beta^2}{4} b \sum_{i=1}^n a_i m_i & \end{aligned}$$

where in the third line we have made use again of Eq.(28). For the second term of Eq.(9), using Eq.(54) and the invariance property (13) we have

$$\begin{aligned} \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} w(\alpha) \exp[-\beta K(\alpha)] \right] &= \quad (60) \\ \frac{1}{N} \mathbb{E} \left[ \ln \sum_{\alpha} w(\alpha) \right] + \beta^2 \frac{(b-1)}{4} \sum_{i=1}^n a_i m_i & \end{aligned}$$

Putting together Eq.(59) and (60) we arrive at the upper bound stated in the Proposition.

□

**Remark 6** *In the following, without loss of generality, we make the assumption*

$$\frac{\kappa_1}{a_1} < \frac{\kappa_2}{a_2} < \dots < \frac{\kappa_n}{a_n} \quad (61)$$

*in order to have a totally nondegenerate sequence of transition temperatures. To express the inequality of (56) in a more transparent form it is convenient to introduce a succession of critical temperatures: for  $i = 1, \dots, n$  let  $\beta_i^* = \beta_c \sqrt{\frac{\kappa_i}{a_i}}$  (where  $\beta_c = 2\sqrt{\ln(2)}$  as in the REM). Under the condition (61), this implies  $\beta_1^* < \dots < \beta_n^*$ . Because of the constraint  $0 < m_1 \leq \dots \leq m_n < 1$  the optimal  $m_i$  is*

$$m_i = \min\{1, \beta_i^*/\beta\} \quad (62)$$

*for  $i = 1, \dots, n$ , the value 1 being attained by taking  $m_i \uparrow 1$  in the infimum of Eq. (56).*

*Therefore, one has*

$$P_N^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) \leq \mathcal{Q}^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}), \quad (63)$$

*where*

$$\mathcal{Q}^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) = \begin{cases} \sum_{k=1}^n \frac{1}{4} a_k (\beta^2 + (\beta_k^*)^2) & \text{for } \beta < \beta_1^* \\ \sum_{k=1}^i \frac{1}{2} a_k \beta \beta_k^* + \sum_{k=i+1}^n \frac{1}{4} a_k (\beta^2 + (\beta_k^*)^2) & \text{for } \beta_i^* \leq \beta \leq \beta_{i+1}^* \\ \sum_{k=1}^n \frac{1}{2} a_k \beta \beta_k^* & \text{for } \beta \geq \beta_n^* \end{cases} \quad (64)$$

### 3.3 Lower Bound

Let us denote the REM pressure (Eq. (5)) as  $P_N^{(1)}(\beta)$ , and its thermodynamic limit (Eq. (6)) as  $P^{(1)}(\beta)$ . This is not really an abuse of notation because if  $n = 1$  then the GREM is the REM, and  $a_1 = \kappa_1 = 1$ . In the same way, we write  $\mathcal{Q}^{(1)}(\beta) = \mathcal{Q}(\beta)$ , where  $\mathcal{Q}(\beta)$  is defined in Eq. (33).

We first prove the following high-temperature result:

**Proposition 3.3** *For  $\beta < \beta_1^*$ ,*

$$P^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) \geq \mathcal{Q}^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}). \quad (65)$$

**Proof:** The proof will follow if we show that

$$P_N^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) \geq \sum_{i=1}^n \kappa_i P_{\kappa_i}^{(1)}(\sqrt{a_i/\kappa_i} \beta), \quad (66)$$

Indeed, taking the thermodynamic limit  $N \rightarrow \infty$  on both sides and using

$$\mathcal{Q}^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) = \sum_{i=1}^n \kappa_i \mathcal{Q}^{(1)}(\sqrt{a_i/\kappa_i} \beta) \quad \text{for } \beta < \beta_1^*. \quad (67)$$

we obtain the Lemma statement. To prove (66) we introduce the interpolating pressure

$$\frac{1}{N} \mathbb{E} \ln \sum_{\sigma \in \Sigma_N} e^{-\beta \tilde{H}(\sigma, t)} \quad (68)$$

with

$$\tilde{H}(\sigma, t) = \sqrt{t} \left[ \sqrt{\frac{N}{2}} \sum_{i=1}^n \sqrt{a_i} X(\pi_1(\sigma), \dots, \pi_i(\sigma)) \right] + \sqrt{1-t} \left[ \sqrt{\frac{N}{2}} \sum_{i=1}^n \sqrt{a_i} Y(\pi_i(\sigma)) \right] \quad (69)$$

where the  $X$ 's and  $Y$ 's are families of i.i.d. Gaussian random variables, each independent from the other. A straightforward differentiation of Eq. (68) combined with integration by parts yields Eq. (66). □

Using this proposition, in conjunction with Proposition 3.2, we actually determine that

$$P^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) = \mathcal{Q}^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}),$$

for  $\beta < \beta_1^*$ . But, as with the REM, this purely high-temperature result also controls the low-temperature behavior using convexity.

**Corollary 3.4** *Suppose that for some  $i \in \{0, \dots, n\}$*

$$0 \leq \sqrt{\frac{a_n}{\kappa_n}} \beta \leq \dots \leq \sqrt{\frac{a_{i+1}}{\kappa_{i+1}}} \beta < \beta_c, \quad \text{and} \quad \sqrt{\frac{a_j}{\kappa_j}} \beta \geq \beta_c \quad \text{for all } j = 1, \dots, i.$$

*Then*

$$P^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa}) \geq \sum_{j=1}^i \frac{a_j \beta_j \beta_j^*}{2} + \sum_{j=i+1}^n \frac{a_i}{4} [\beta_i^2 + (\beta_i^*)^2].$$

**Proof:** This corollary is proved by induction on  $i$ . For  $i = 0$ , it is the content of Proposition 3.3. For each step of the induction argument, the proof uses convexity in exactly the same way as the proof of Corollary 2.4. Specifically, in the present case, one will notice that  $P_N^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa})$  is jointly continuous in the variables  $\sqrt{a_i}\beta$  for  $i = 1, \dots, n$ . So one can think of keeping  $\beta$  fixed, and varying  $\sqrt{a_1}, \dots, \sqrt{a_n}$ . With this comment, the rest of the proof is an easy exercise, and we leave it to the interested reader.  $\square$

One can see, by looking at Remark 6, that this corollary gives a quantity for the lower bound for  $P^{(n)}(\beta; \mathbf{a}, \boldsymbol{\kappa})$  which matches the upper bound of Proposition 3.2, in all temperature ranges. Therefore, the simple upper bound of Proposition 3.2 is actually the sharpest possible.

## A Proof of Lemma 2.6: Concentration of measure

In this Appendix we prove the “concentration of measure” inequality (Lemma 2.6). The proof is a generalization of the proof for the SK model (Corollary 2.2.5 of [24]). We show that the standard deviation inequality apply to a large class of Gaussian spin-glass models, which includes both mean-field (SK,p-spin,REM,GREM) and finite dimensional models (Edwards-Anderson, Random Field). We recall the basic result for a function of Gaussian variables:

**Theorem A.1 (Talagrand)** *Consider a Lipschitz function  $F$  on  $\mathbb{R}^M$ , of Lipschitz constant  $A$ . If  $J_1, \dots, J_M$  are independent, standard normal random variables, then for each  $t > 0$ ,*

$$\mathbb{P}\{|F(\mathbf{J}) - \mathbb{E}[F(\mathbf{J})]|\geq At\} \leq 2e^{-t^2/4}. \quad (70)$$

This is Theorem 2.2.4 of [24]. Talagrand proves this using the “smart path” method, which is another interpolation argument. We will not repeat the proof here.

Let us consider the general Hamiltonian given by

$$H_\Lambda(\sigma; \mathbf{J}) = -\frac{1}{\sqrt{2}} \sum_{X \in \Lambda} \Delta_X J_X \sigma_X, \quad (71)$$

where  $\Lambda \subset \mathbb{Z}^d$ ,  $\sigma_X = \prod_{i \in X} \sigma_i$ ,  $\{\Delta_X\}_{X \in \Lambda} \geq 0$  and the  $\{J_X\}_{X \in \Lambda}$  are a family of i.i.d. standard Gaussian random variables,  $\mathbb{E}[J_X] = 0$ ,  $\mathbb{E}[J_X J_Y] = \delta_{X,Y}$ . A sufficient condition for the existence of thermodynamic limit [9] is that there exist a constant  $c < \infty$  such that

$$\sup_{\Lambda \subset \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{X \subset \Lambda} \Delta_X^2 \leq c \quad (72)$$

The same condition also implies the validity of the Ghirlanda-Guerra identities [8, 14].

To prove Lemma 2.6, we need to show that the random pressure function

$$P_\Lambda(\mathbf{J}) = \frac{1}{|\Lambda|} \ln \sum_{\sigma \in \Sigma_N} e^{-\beta H_\Lambda(\sigma; \mathbf{J})} \quad (73)$$

is Lipschitz. For this, we note that

$$P_\Lambda(\mathbf{J}) - P_\Lambda(\mathbf{J}') = \int_0^1 \frac{dP_\Lambda(t\mathbf{J} + (1-t)\mathbf{J}')}{dt} dt. \quad (74)$$

On the other hand

$$\frac{dP_\Lambda(t\mathbf{J} + (1-t)\mathbf{J}')}{dt} = \frac{\beta}{\sqrt{2}|\Lambda|} \sum_{X \subset \Lambda} \omega_t(\sigma_X) \Delta_X (J_X - J'_X) \quad (75)$$

with

$$\omega_t(\sigma_X) = \frac{\sum_{\sigma} \sigma_X e^{-\beta H_\Lambda(\sigma; t\mathbf{J} + (1-t)\mathbf{J}')}}{\sum_{\sigma} e^{-\beta H_\Lambda(\sigma; t\mathbf{J} + (1-t)\mathbf{J}')}} \quad (76)$$

From Eq. (74), by using  $|\sigma_X| \leq 1$ , Cauchy-Schwarz inequality and the thermodynamic stability condition (72), it then follows

$$|P_\Lambda(\mathbf{J}) - P_\Lambda(\mathbf{J}')| \leq \beta \sqrt{\frac{c}{2|\Lambda|}} \|\mathbf{J} - \mathbf{J}'\|. \quad (77)$$

where  $\|\cdot\|$  denotes the  $L_2$ -norm. Therefore,  $P_\Lambda(\mathbf{J})$  is Lipschitz, with Lipschitz constant  $A = \beta \sqrt{\frac{c}{2|\Lambda|}}$ . Applying Theorem A.1 it gives

$$\mathbb{P}\{|P_\Lambda(\mathbf{J}) - \mathbb{E}[P_\Lambda(\mathbf{J})]| \geq t\} \leq 2 \exp\left(-\frac{t^2 |\Lambda|}{2c\beta^2}\right). \quad (78)$$

This result apply equally well to all general Hamiltonian of the form (71). The REM model is obtained from Eq. (71) with the choice  $|\Lambda| = N$ ,  $\Delta_X = \sqrt{N2^{-N}}$  (this is indeed Hamiltonian (3)). In this case the condition (72) gives  $c = 1$  and the statement of Lemma 2.6 is proved.

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